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MATH 260R

Conditional Expectation.

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N}$ a sub-σ algebra of $\mathcal{M}$, and $\nu = \mu|\mathcal{N}$. Given any $f \in L^1(\mu)$, we want to show the following hold: Theorem:

There exists $g \in L^1(\nu)$ such that $\int_E f\,d\mu = \int_E g\,d\nu$ for all $E \in \mathcal{N}$. If g' is another such function then $g = g'$ ν-a.e. Proof:

We define a signed measure $d\lambda = f\,d\mu$ on $\mathcal{N}$.

As $|\lambda(X)| = |\int_X f\,d\mu| \leq \int_X |f|\,d\mu < \infty$, λ is finite and hence σ-finite.

Note when $E \in \mathcal{N}$ such that $\nu(E) = 0$, then $\mu(E) = \nu(E) = 0$, so $\lambda(E) = \int_E f\,d\mu = 0$. It follows that $\lambda \ll \nu$. By the Radon-Nikodym theorem, it follows that $d\lambda = g\,d\nu$ for some $g \in L^1(\nu)$, then $f\,d\mu = d\lambda = g\,d\nu$, so $\int_E f\,d\mu = \int_E g\,d\nu$ for all $E \in \mathcal{N}$.

Now, let $g' \in L^1(\nu)$ also satisfy $\int_E f\,d\mu = \int_E g'\,d\nu$ for all $E \in \mathcal{N}$. We want to show $g = g'$ ν.a.e., which is the same as showing $\nu(\{x \in X \mid g(x) \neq g'(x)\}) = 0$

Note: $\{x \in X \mid g(x) \neq g'(x)\}$

$$= \{x \in X \mid g(x) > g'(x)\} \sqcup \{x \in X \mid g(x) < g'(x)\},$$

and:

$$\{x \in X \mid g(x) > g'(x)\} = \{x \in X \mid g(x) - g'(x) > 0\}$$

$$= \bigcup_{n=1}^{\infty} \{x \in X \mid g(x) - g'(x) \geq \tfrac{1}{n}\}$$

For a fixed $n \in \mathbb{N}$, we show $\nu(\{x \in X \mid g(x) - g'(x) \geq \frac{1}{n}\}) = 0$.

Clearly $E_n = \{x \in X \mid g(x) - g'(x) \geq \frac{1}{n}\} \in N$, and so

$$\int_{E_n} g - g' \, d\nu = \int_{E_n} g \, d\nu - \int_{E_n} g' \, d\nu$$

$$= \int_{E_n} f \, d\mu - \int_{E_n} f \, d\mu$$

$$= 0.$$

But $\int_{E_n} g - g' \, d\nu \geq \int_{E_n} \frac{1}{n} \, d\nu = \frac{1}{n} \cdot \nu(E_n) \geq 0$,

so $\frac{1}{n} \nu(E_n) = 0$, $\nu(E_n) = 0$ for all n, so by the subadditivity of measures:

$$0 \leq \nu(\{x \in X \mid g(x) > g'(x)\})$$

$$= \nu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$\leq \sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} 0 = 0,$$

so $\nu(\{x \in X \mid g(x) > g'(x)\}) = 0$.

By interchanging the positions of g and g' in the above argument, we get $\nu(\{x \in X \mid g'(x) > g(x)\}) = 0$.

Then:

$$\nu(\{x \in X \mid g(x) \neq g'(x)\}) =$$

$$\nu(\{x \in X \mid g(x) > g'(x)\}) + \nu(\{x \in X \mid g(x) < g'(x)\}) = 0.$$

This shows g is unique ν-a.e.

In probability theory, g is called the conditional expectation of f on $\mathcal{N}$, and is denoted $g = E(f|\mathcal{N})$

Simple Application in Probability:

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X be a random variable (measurable function), so that its expectation $E(X) = \int_\Omega X\,dP$. We claim that if $E(X^2) < \infty$, then $E(X|\mathcal{F})$ is the variable $Y \in \mathcal{F}$ that minimizes the "mean square error" $E(X-Y)^2$.

We need the following properties:

① $E(E(X|\mathcal{F})) = E(X)$; ← write these as integrals

② If $X \in \mathcal{F}$, and $E|Y|, E|XY| < \infty$, then $E(XY|\mathcal{F}) = XE(Y|\mathcal{F})$.

Note if $Z \in L^2(\mathcal{F})$, then by ②, we have:

$Z\,E(X|\mathcal{F}) = E(ZX|\mathcal{F})$. ($E|XZ| < \infty$ by Cauchy-Schwartz)

Taking expected values:

$E(ZE(X|\mathcal{F})) = E(ZX|\mathcal{F}) = E(ZX)$, that is:

$E(Z(X - E(X|\mathcal{F}))) = 0$ for $Z \in L^2(\mathcal{F})$.

If $Y \in L^2(\mathcal{F})$, and $Z = E(X|\mathcal{F}) - Y$, then:

$E(X-Y)^2 = E(X - E(X|\mathcal{F}) + Z)^2$

$= E(X - E(X|\mathcal{F}))^2 + 2E((X - E(X|\mathcal{F}))Z) + EZ^2$

$= E(X - E(X|\mathcal{F}))^2 + EZ^2$

$E(X-Y)^2$ is minimized when $Z = 0$, that is, $Y = E(X|\mathcal{F})$.