Key

Backup notes if people are curious.

Ideally, write on board before presentation begins.

Usual pacing: 10 minutes / page.
⇒ ~25 minutes

The Gauss Green Theorem

The graph of $u : \mathbb{R}^{n-1} \to \mathbb{R}$ over $G \subseteq \mathbb{R}^{n-1}$ is

$$\Gamma(u; G) = \{ (x, u(x)) : x \in G \}.$$

For brevity, set $\Gamma(u) = \Gamma(u, \mathbb{R}^{n-1})$.

Thm 9.1 : For every $\varphi \in C_c^0(\mathbb{R}^n)$,

$$\int_{\Gamma(u)} \varphi \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^{n-1}} \varphi(z, u(z)) \sqrt{1 + |\nabla u(z)|^2} \, dz.$$

To prove thm. 9.1, we use ...

(Recall) Remark 8.3 : If $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is an injective Lipshitz function and $\varphi \in C_c^0(\mathbb{R}^n)$,

$$\int_{f(\mathbb{R}^{n-1})} \varphi \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^{n-1}} \varphi(f(z)) \, Jf(z) \, dz.$$

Proof 9.1 : Define $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ by $f(z) = (z, u(z))$, and note f is injective and Lipschitz.
We calculate the Jacobian of f and invoke theorem 8.1 and remark 8.3 to obtain the result.

$\square$

Thm 9.3 (Gauss Green): If E is an open set with C^1-boundary, then $\forall \varphi \in C_c^1(\mathbb{R}^n)$,

$$\int_E \nabla \varphi(x)\, dx = \int_{\partial E} \varphi\, \nu_E \, d\mathcal{H}^{n-1}.$$

Notation: Define $p: \mathbb{R}^n \to \mathbb{R}^{n-1}$ by $p(y_1, \dots, y_n) = (y_1, \dots, y_{n-1})$.
For $x \in \mathbb{R}^n$, $s > 0$,

$$\mathcal{C}(x,s) = \{ y \in \mathbb{R}^n : \| px - py \| < s,\ |x_n - y_n| < s \}.$$

Proof: Step 1: Fix $x \in \partial E$.
There exists $s > 0$, $u \in C^1(B(px, s))$ s.t.

1. $\mathcal{C}(x,s) \cap E = \{ y \in \mathcal{C}(x,s) : y_n > u(py) \}$.
2. $\mathcal{C}(x,s) \cap \partial E = \{ y \in \mathcal{C}(x,s) : y_n = u(py) \}$.

For $\delta > 0$, define the Lipschitz function $f_\delta : \mathcal{C}(x,s) \to \mathbb{R}$ by

$$f_\delta(y) = \begin{cases} 1 & y_n > u(py) + \delta \\ 0 & y_n < u(py) - \delta \\ \frac{1}{2\delta}\left(y_n - u(py) + \delta \right) & |u(py) - y_n| < \delta. \end{cases}$$

For $\varphi \in C_c^1(\mathcal{C}(x,s))$,

$$\int_E \nabla \varphi = \int_{E \cap \mathcal{C}(x,s)} \nabla \varphi$$

$$\overset{DCT}{=} \lim_{\delta \to 0^+} \int_{\mathcal{C}(x,s)} f_\delta \nabla \varphi \qquad // \ f_\delta \to \mathbb{1}_{\mathcal{C}(x,s) \cap E}$$

$$\overset{IBP}{=} - \lim_{\delta \to 0^+} \int_{\mathcal{C}(x,s)} \varphi \nabla f_\delta.$$

Skipping some steps,

$$-\lim_{\delta\to 0^+}\int_{C(x,s)} \varphi \nabla f_\delta$$

unit normal

$$\overset{\star\star}{=} \int_{B(px,s)} \varphi(z,u(z))\, \nu_E(z,u(z)) \sqrt{1+|\nabla u(z)|^2}\, dz$$

$$\overset{\star}{\underset{\text{Thm 9.1}}{=}} \int_{C(x,s)\cap\partial E} \varphi \nu_E \, d\mathcal{H}^{n-1}$$

$$= \int_{\partial E} \varphi \nu_E \, d\mathcal{H}^{n-1}$$

Step 2: Fix $\varphi \in C_c^1(\mathbb{R}^n)$.
Let A be an open set s.t. $\operatorname{spt}\varphi \cap \partial E \subseteq A$.
We know

$$\{B(x,s) \subseteq A : x \in A \cap \partial E,\ s \text{ as in step } 1\}.$$

is an open cover of $\operatorname{spt}\varphi \cap \partial E$.
Let $\{B(x_k,s_k)\}_{k=1}^N$ be a finite subcover.
By step 1, $\forall\, 1 \le k \le N$, $\xi \in C_c^1(B(x_k,s_k))$,

$$\int_E \nabla(\xi\varphi) = \int_{\partial E} \xi \varphi \nu_E \, d\mathcal{H}^{n-1}.$$

A standard partition of unity argument gives the result.

$\square$

$\bigstar$: $\mathcal{C}(x,s) \cap \partial E = \Gamma(u, B(px, s))$.

$\subseteq$: If $y \in \mathcal{C}(x,s) \cap \partial E$,

1. $|py - px| < s$.
2. $y_n = u(py)$.

So $y \in \Gamma(u, B(px, s))$.

$\supseteq$: If $y \in \Gamma(u, B(px,s))$, then

1. $|py - px| < s$.
2. $y_n = u(py)$.

We NTS $|y_n - x_n| < s$, i.e. $|u(py) - u(px)| < s$.
This follows by the implicit function thm.
See remark 9.2

Write on board

$\bigstar\bigstar$: Take gradient of f, take $\delta \to 0$, use

$$\nu_E(y) = \frac{(\nabla u(py), -1)}{\sqrt{1 + |\nabla u(py)|^2}} \quad \forall y \in \mathcal{C}(x,s) \cap \partial E.$$