HAUSDORFF MEASURES AND HAUSDORFF DIMENSION

Given $E \subset \mathbb{R}^n$, with $k, n \in \mathbb{N}$, and some fixed $\delta \in (0, \infty]$. Let $\mathcal{F}$ be a countable covering of E such that for each $F \in \mathcal{F}$, $\operatorname{diam}(F) < \delta$. We define the **$k$-dimensional Hausdorff measure of step δ** of E to be

$$\mathcal{H}^k_\delta(E) := \inf_{\mathcal{F}} \left\{ \sum_{F \in \mathcal{F}} \omega_k \left(\frac{\operatorname{diam}(F)}{2} \right)^k \right\}$$

where $\omega_k := \mathcal{L}^k(B^k_1(0))$, the Lebesgue measure of the k-dimensional unit ball. We define the **k-dimensional Hausdorff measure** of E to be

$$\mathcal{H}^k(E) := \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E)$$

Indeed, both $\mathcal{H}^k$ and $\mathcal{H}^k_\delta$ are outer measures, obey the scaling law (i.e. $\lambda^k \mathcal{H}^k(E) = \mathcal{H}^k(\lambda E)$), and are translation invariant (i.e. $\mathcal{H}^k(x + E) = \mathcal{H}^k(E)$).

What if $k \notin \mathbb{N}$? Every element of our previous formula should hold, with the exception of how we conceived ω_k; it does not hold by virtue of the Lebesgue measure not being defined for a non-discrete dimension. So we instead define ω_s for $s \in [0, \infty) \subset \mathbb{R}$ as

$$\omega_s := \frac{\pi^{s/2}}{\Gamma(1 + \frac{s}{2})}$$

(where Γ is the gamma function) and the **s-dimensional Hausdorff measure** is thus, being nearly identical to the k-dimensional case, defined as

$$\mathcal{H}^s(E) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E) := \lim_{\delta \to 0^+} \inf_{\mathcal{F}} \left\{ \sum_{F \in \mathcal{F}} \omega_s \left(\frac{\operatorname{diam}(F)}{2} \right)^s : \operatorname{diam}(F) < \delta \right\}$$

which is hereafter to be the object of our attention to serve the end of generality.

We have a vague understanding of what relates a three-dimensional surface to $\mathbb{R}^2$, or a two- or three-dimensional curve to $\mathbb{R}$, in that one may parametrise such submanifolds to these lower dimensional Euclidean spaces. To codify this idea better, we introduce the notion of Hausdorff dimension.

We define the **Hausdorff dimension** of a set $E \subset \mathbb{R}^n$ as

$$\dim E := \inf_{s \in [0, \infty)} \{ s : \mathcal{H}^s(E) = 0 \}$$

Let us list and prove three properties of the Hausdorff dimension.

Proposition 1. *$\mathcal{H}^s(\mathbb{R}^n) = 0$ for each $s > n$.*

Proof. Let $s > n$. Consider $Q := (0,1)^n$. Notice that due to the scaling law, we have

$$\lambda^n \mathcal{H}^s(E) = \mathcal{H}^s(\lambda E) \xrightarrow{\lambda\to\infty} \mathcal{H}^s(\mathbb{R}^n)$$

so it suffices to show that $\mathcal{H}^s(Q) = 0$. Partition Q into k^n many cubes. The diagonal (and hence diameter) of each cube is then $\sqrt{n}/k$. We have

$$\mathcal{H}^s_{\sqrt{n}/k}(Q) \leq k^n \omega_s \left(\frac{\sqrt{n}}{2k}\right) = \frac{k^n}{k^s}\omega_s \left(\frac{\sqrt{n}}{2}\right)^s$$

Now take $k \to \infty$, and we get $\mathcal{H}^s(Q) = 0$. ■

It also follows from proposition 1 that $\dim E \in [0, n]$ whenever $E \subset \mathbb{R}^n$.

Proposition 2. Given $E \subset \mathbb{R}^n$. $\mathcal{H}^s(E) = \infty$ if $s < \dim E$.

Proof. Assume for the sake of contradiction that there exists some $s < \dim E$ such that $\mathcal{H}^s(E) < \infty$. Now given any $t > s$ and $\delta > 0$, let $\mathcal{F}_\delta$ be a countable covering of E such that for each $F \in \mathcal{F}$, $\operatorname{diam}(F) < \delta$. Then we have

$$\begin{aligned}\mathcal{H}^t_\delta(E) \leq \omega_t \sum_{F\in\mathcal{F}} \left(\frac{\operatorname{diam}(F)}{2}\right)^t &\leq \frac{\omega_t}{\omega_s}\omega_s \sum_{F\in\mathcal{F}} \left(\frac{\operatorname{diam}(F)}{2}\right)^{t-s} \left(\frac{\operatorname{diam}(F)}{2}\right)^s \\ &\leq \frac{\omega_t}{\omega_s}\left(\frac{\delta}{2}\right)^{t-s} \mathcal{H}^s(E)\end{aligned}$$

Now take $\delta \to 0^+$ and we get $\mathcal{H}^t(E) = 0$, for *any* $t > s$. Then $\dim E \leq s < t$ which contradicts our assumption $s < \dim E$. ■

(part below could not be covered in the presentation)

Proposition 3. $\mathcal{H}^0(E) = \#(E)$ ($\mathcal{H}^0$ is the counting measure).

Proof. Exercise to the reader.

Proposition 4. Given $E \subset \mathbb{R}^n$. If $\mathcal{H}^s_\infty(E) = 0$, then $\mathcal{H}^s(E) = 0$

Proof. If $s = 0$, the conclusion naturally follows from proposition 3. So let $s > 0$. Since $\mathcal{H}^s_\infty(E) = 0$, we have that for any given $\epsilon > 0$,

$$\omega_s \sum_{F\in\mathcal{F}} \left(\frac{\operatorname{diam}(F)}{2}\right)^s \leq \epsilon$$

for some countable covering $\mathcal{F}$ of E. Now for any given $F \in \mathcal{F}$

$$\Rightarrow \omega_s \left(\frac{\operatorname{diam}(F)}{2}\right)^s \leq \varepsilon$$

$$\Rightarrow \operatorname{diam}(F) \leq 2\left(\frac{\epsilon}{\omega_s}\right)^{1/s}$$

giving

$$\sup \operatorname{diam}(F) \leq 2\left(\frac{\epsilon}{\omega_s}\right)^{1/s} =: \delta(\varepsilon)$$

Taking right hand side as our step, we have

$$\mathcal{H}^s_{\delta(\varepsilon)} \leq \varepsilon$$

Now if we take $\varepsilon \to 0$ we get $\delta(\varepsilon) \to 0$, which then gives us $\mathcal{H}^s(E) = 0$. ■

Reference. Maggi F. *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory.* Cambridge University Press; 2012.