# Hausdorff = Lebesgue, i.e. $\mathcal{H}^n = |\cdot|$ (in $\mathbb{R}^n$)

Theorem: If $E \subset \mathbb{R}^n$ and $\delta \in (0,\infty]$, then

$$|E| = H^n(E) = H^n_\delta(E).$$

Recall: $|E|$ can be defined as $\inf_{\mathcal{F}} \sum_{Q \in \mathcal{F}} \mathrm{vol}(Q)$, $Q$ cubes and $\mathcal{F}$ a cover of $E$.

i) Vitali's Property: If $A \subset \mathbb{R}^n$ is open and $\delta > 0$, $\exists\, \mathcal{F}$ disjoint family of closed balls contained in $A$ s.t. $\left| A - \bigcup_{\overline{B} \in F} \overline{B} \right| = 0$ and $\operatorname{diam} \overline{B} < \delta$ for all $B \in \mathcal{F}$. Call $\mathcal{F}$ a Vitali family for $A, \delta$.

ii) Isodiametric Ineq: For all $E \subset \mathbb{R}^n$, $|E| \leq \omega_n \left( \frac{\operatorname{diam}(E)}{2} \right)^n$.

<u>Pf of Thm:</u>

1) Claim: $\omega_n \left( \frac{\sqrt{n}}{2} \right)^n |E| \geq H^n_\infty(E)$

$\rightarrow$ If $\mathcal{F}$ is a "competitor" in the definition of $|E|$ and $r(F)$ denotes the side length of a cube $F \in \mathcal{F}$, then $\operatorname{diam}(F) = \sqrt{n}\, r(F)$ and so

$$H^n_\infty(E) \underset{\text{iso}}{\leq} \omega_n \sum_{F \in \mathcal{F}} \left( \frac{\operatorname{diam}(F)}{2} \right)^n = \omega_n \left( \frac{\sqrt{n}}{2} \right)^n \sum_{F \in \mathcal{F}} r(F)^n.$$

As $\mathcal{F}$ was an arbitrary (valid) cover, the claim follows at once.

2) Claim: $|E| \geq \mathcal{H}^n(E)$

$\rightarrow$ Assume $|E| < \infty$. Fix $\varepsilon, \delta > 0$ and let $A \supset E$ be open s.t. $|A| < |E| + \varepsilon$ and take some Vitali family $\mathcal{F}$ for $A, \delta$. Write $F = \bigcup_{\bar{B} \in \mathcal{F}} \bar{B}$.

Then we have the following:

$$(\star) \quad \mathcal{H}^n_\delta(F) \leq \omega_n \sum_{\bar{B} \in \mathcal{F}} \left(\frac{\operatorname{diam} \bar{B}}{2}\right)^n = \sum_{\bar{B} \in \mathcal{F}} |\bar{B}| = \Big|\bigcup_{\bar{B} \in \mathcal{F}} \bar{B}\Big| = \underbrace{|A|}_{\leq |E| + \varepsilon}$$

By the first claim, $\mathcal{H}^n_\infty(A - F) = 0$. Then in particular $\mathcal{H}^n_\infty(E - F) = 0 \overset{*}{\Longrightarrow} \mathcal{H}^n_\delta(E - F) = 0$.

$*$: Clear for $n = 0$. For $n > 0$: since $\mathcal{H}^n_\infty(E) = 0$, $\forall \varepsilon > 0$ $\exists \mathcal{F}$ countable cover with $\omega_n \sum \left(\frac{\operatorname{diam}(F)}{2}\right)^n \leq \varepsilon$ so that $\sup_F \operatorname{diam}(F) \leq 2\left(\frac{\varepsilon}{\omega_n}\right)^{1/n} = \delta(\varepsilon)$. Then $\mathcal{H}^n_{\delta(\varepsilon)}(E) \leq \varepsilon$ with $\delta(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$.

Then $\mathcal{H}^n_\delta(E) \leq \mathcal{H}^n_\delta(E \cap F) + \mathcal{H}^n_\delta(E - F) \leq \mathcal{H}^n_\delta(F) \leq |E| + \varepsilon$.
Send $\varepsilon, \delta \to 0$.

The other inequalities $\left(|E| \leq \mathcal{H}^n_\delta(E) \leq \mathcal{H}^n(E)\right)$ follow as such: Given $\delta \in (0, \infty]$ and $\mathcal{F}_\delta$ a cover of $E$ by sets $F$ with $\operatorname{diam}(F) \leq \delta$, the isodiametric inequality gives us

$$|E| \leq \Big|\bigcup_{F \in \mathcal{F}_\delta} F\Big| \leq \sum_{F \in \mathcal{F}_\delta} |F| \leq \omega_n \sum_{F \in \mathcal{F}_\delta} \left(\frac{\operatorname{diam}(F)}{2}\right)^n. \qquad \square$$