Isodiametric Inequality and Steiner Symmetrization

One of the major properties of the Hausdorff measure is that it that is corresponds to the Lebesgue measure. To show that the Hausdorff measure equals to the Lebesgue we use two tools, the first being a variant of the Vitali property and the second one being the Isodiametric inequality.

Proposition (Isodiametric Inequality). *For any Lebesgue measurable set $E \subseteq \mathbb{R}^n$,*

$$|E| \leq \omega_n \left(\frac{\operatorname{diam}(E)}{2}\right)^n$$

where ω_n is the volume of the unit n-ball. In other words, among all sets with fixed diameter, balls have the maximum volume.

One naive approach is hoping that E is contained in some ball of radius $\frac{\operatorname{diam}(E)}{2}$ because if $E \subseteq B_{\frac{\operatorname{diam}(E)}{2}}(x)$ for some $x \in \mathbb{R}^n$, the proposition holds immediately by monotonicity. Unfortunately is not always true with one quick counterexample being the equilateral triangle.

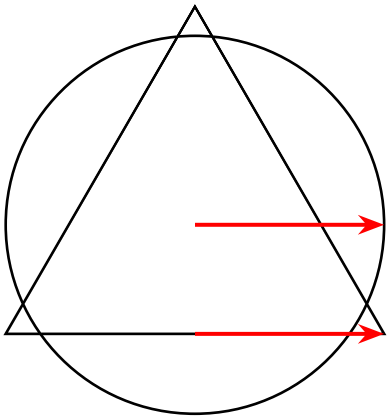

Figure 1: Counterexample of the equilateral triangle and its circle.

Although the idea does not hold for every set in $\mathbb{R}^n$, we can consider a measure preserving transformation where the set is contained in the ball. More precisely, we construct a map $E \mapsto E'$ such that $|E| = |E'|$ but $E' \subseteq B_{\frac{\operatorname{diam}(E)}{2}}(0)$. This is resolved using *Steiner symmetrization.*

Definition (Steiner Symmetrization). Let $E \subseteq \mathbb{R}^n$ and for each $x \in \mathbb{R}^n$ write $x = (z, t)$ where $z \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$. For each $z \in \mathbb{R}^{n-1}$, let

$$E_z = \{t \in \mathbb{R} : (z, t) \in E\}$$

so E_z denotes the cross sections of E at z. Then the *Steiner symmetrization* E^s of E is defined as

$$E^s = \left\{(z, t) \in \mathbb{R}^n : |t| \leq \frac{\mathcal{L}^1(E_z)}{2}\right\}.$$

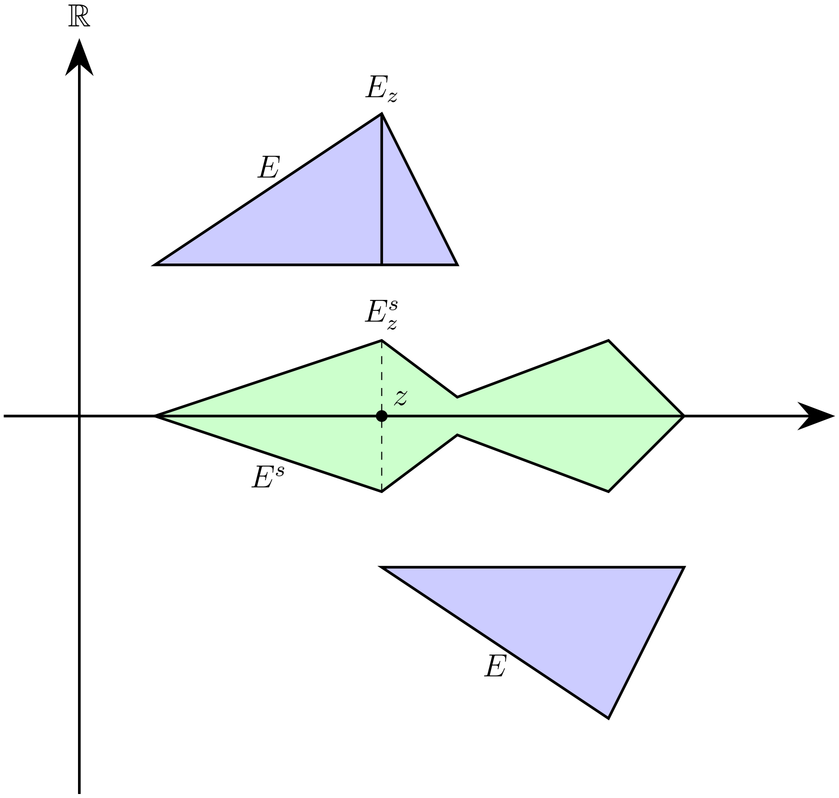

Figure 2: Steiner symmetrization.

If E is Lebesgue measurable, then E_z is Lebesgue measurable so by Fubini's theorem

$$|E| = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E_z)dz = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E_z^s)dz = |E^s|$$

so we have that Steiner symmetrization is measure preserving.

Claim. $\operatorname{diam}(E^s) \leq \operatorname{diam}(E)$.

Proof. First let $p\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $q\colon \mathbb{R}^n \to \mathbb{R}$ be the projections where $p(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$ and $q(x_1, \dots, x_n) = x_n$ respectively. Now fix $x \in E^s$ and choose $m(x)$, $M(x) \in \overline{E}$ such that

$$\begin{aligned} &pm(x) = px = pM(x) \\ &qm(x) \leq qz \leq qM(x) && \text{for each } z \in E \text{ where } pz = px. \end{aligned}$$

Then for any $x, y \in E^s$ by construction of E^s and choice of $m(x)$ and $M(x)$,

$$|qx - qy| \leq \max\{|qM(x) - qm(y)|, |qm(x) - qM(y)|\}.$$

Since $|z|^2 = |pz|^2 + |qz|^2$,

$$|x - y| \leq \max\{|M(x) - m(y)|, |m(x) - M(y)|\} \leq \operatorname{diam}(\overline{E}) = \operatorname{diam}(E).$$

Since x, y is arbitrary, the claim follows. ■

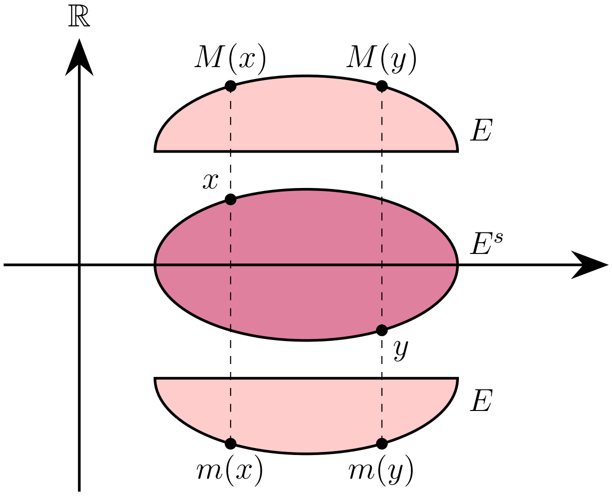

Figure 3: Proof of Claim.

Remark. One can show that $P(E^s) \leq P(E)$ where if ∂E is C^1, $P(E) = \mathcal{H}^{n-1}(\partial E)$.

Proof of Isodiametric Inequality. Consider $\overline{E}$ since $|E| \leq \left|\overline{E}\right|$ and $\operatorname{diam}(E) = \operatorname{diam}\left(\overline{E}\right)$. For any Lebesgue mesureable set F, let F^i denote the Steiner Symmetrization of F with respect to the i-th coordinate axis. Now let $E_0 = \overline{E}$, and for each $1 \leq i \leq n$, let $E_i = (E_{i-1})^i$. By construction, since each E_i is reflective along the i-th coordinate axis, E_n is symmetric under reflection so $x \in E_n$ if and only if $-x \in E_n$. Thus $E_n \subseteq B_{\frac{\operatorname{diam}(E_n)}{2}}$ so

$$|E_n| \leq \omega_n \left(\frac{\operatorname{diam}(E_n)}{2}\right)^n.$$

Now since $\operatorname{diam}(E_n) \leq \operatorname{diam}\left(\overline{E}\right) = \operatorname{diam}(E)$, stringing the inequalities together we get

$$|E| \leq |E_n| \leq \omega_n \left(\frac{\operatorname{diam}(E_n)}{2}\right)^n \leq \omega \left(\frac{\operatorname{diam}(E)}{2}\right)^n$$

giving us the desired inequality. ■