Lecture 10

260R, Advanced Measure Theory

Reminder: Presentation Topics - May 5
- May 14

Prop: Given $F \in BV$, right cts, $F(-\infty)=0$, then F' exists $\mathcal{L}$-a.e. and $F' \in L^1(\mathcal{L})$.

Furthermore,

$\mu_F \perp \mathcal{L} \iff F' = 0$ $\mathcal{L}$-a.e.

$\mu_F \ll \mathcal{L} \iff F(x) = \underbrace{\int_{-\infty}^{x} F'(t)\, d\mathcal{L}(t)}_{= \int_{-\infty}^{x} d\mu_F}$.

__Def__: $F:\mathbb{R}\to\mathbb{R}$ is absolutely continuous if $\forall\,\varepsilon>0$, $\exists\,\delta>0$ s.t. for all finite collections of disjoint intervals $(a_1,b_1),\dots,(a_N,b_N)$,

$$\sum_{i=1}^{N}|b_i-a_i|<\delta \Rightarrow \sum_{i=1}^{N}|F(b_j)-F(a_i)|<\varepsilon.$$

__Prop__: Given $F\in BV$, right cts, $F(-\infty)$,

F is abs cts $\iff \mu_F \ll \mathcal{L}$.

Pf: Last time, we showed
$\mu_F \ll \mathcal{L} \Rightarrow F$ is abs cts

Now suppose F is abs cts.
Fix $E \in \mathcal{B}_{\mathbb{R}}$ s.t. $\mathcal{L}(E)=0$.
We must show $\mu_F(E)=0$.

Fix $\varepsilon>0$ arbitrary. By
defn of abs cty, $\exists\, \delta$ s.t.

$$\sum_{i=1}^{N} |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^{N} |F(b_j)-F(a_i)| < \varepsilon.$$

By outer regularity $\mu_F^+, \mu_F^-, \mathcal{L}$
$\forall n \in \mathbb{N}$, $\exists\, U_n^+, U_n^-, U^{\mathcal{L}}$ open
sets containing E s.t.

$$\mu_F^+(\mathcal{U}_n^+) \le \mu_F^+(E) + \frac{1}{n}$$
$$\mu_F^-(\mathcal{U}_n^-) \le \mu_F^-(E) + \frac{1}{n}$$
$$\mathcal{L}(\mathcal{U}^{\mathcal{L}}) \le \mathcal{L}(E) + \delta$$

Let $\mathcal{U}_n := \mathcal{U}_n^+ \cap \mathcal{U}_n^- \cap \mathcal{U}^{\mathcal{L}}$.
Define $V_1 = \mathcal{U}_1$, $V_2 = \mathcal{U}_2 \cap V_1$, $V_3 = \mathcal{U}_3 \cap V_2, \dots$

Then $V_1 \supseteq V_2 \supseteq V_3$ are open sets containing E s.t. $\mathcal{L}(V_n) < \delta$ $\forall n$ and

$$\lim_{n\to\infty} \mu_F(V_n) = \lim_{n\to\infty} \mu_F^+(V_n) - \mu_F^-(V_n)$$
$$= \mu_F^+(E) - \mu_F^-(E)$$
$$= \mu_F(E).$$

Since $V_n = \bigsqcup_{k=1}^{\infty} (a_k^n, b_k^n)$, the fact that $\mathcal{L}(V_n) < \delta$, by defn of abs cty, $\forall N \in \mathbb{N}$

$$\sum_{k=1}^{N} |\mu_F((a_k^n, b_k^n))| = \sum_{k=1}^{N} |F(b_k^n) - F(a_k^n)| < \varepsilon$$

$$|\mu_F(V_n)| \leq \sum_{k=1}^{\infty} |\mu_F((a_k^n, b_k^n))| \leq \varepsilon.$$

Then $|\mu_F(E)| = \lim_{n \to \infty} |\mu_F(V_n)| \leq \varepsilon.$

Since ε was arb, this shows $\mu_F(E) = 0$. $\square$

In fact, even if μ_F is <u>not</u> abs cts wrt $\mathcal{L}$, we can still think of μ_F as a weak derivative of F.

<u>Thm</u>: Suppose $F, G \in BV$, F right cts, G cts. Then $\forall -\infty < a < b < +\infty$,

$$\int_{(a,b]} F \, d\mu_G + \int_{(a,b]} G \, d\mu_F = \underline{F(b)G(b)} \underline{- F(a)G(a)}$$

Rmk: WLOG $F(-\infty) = G(-\infty) = 0$, so μ_G, μ_F well defined.

Cor: Given $F \in BV$, right cts and $G \in C_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} F(x) G'(x) \, d\mathcal{L}(x) = -\int_{\mathbb{R}} G(x) \, \underbrace{d\mu_F(x)}_{\text{this is the weak derivative.}}$$

Pf: This follows from thm by noting G is abs cts, since

$$\sum_{i=1}^{N} |G(x_i) - G(x_{i-1})| \overset{\text{mean value theorem}}{\leq} \underbrace{\|G'\|_\infty}_{<+\infty \text{, since } G \text{ is cptly supported}} \sum_{i=1}^{N} |x_i - x_{i-1}|$$

Because G abs cts,

$$d\mu_G = G'(x) \, d\mathcal{L}(x).$$

We also have $G \in BV$,

$$T_F G(+\infty) = \int_{\mathbb{R}} |G'(x)| \, d\mathcal{L}(x) < +\infty$$

$\square$

We now return to the proof of the integration by parts theorem.

Pf: It suffices to prove the result replacing the BV hypothesis with increasing.

The general case then follows, since F & G may be expressed as differences of such fns.

Now μ_F, μ_G positive finite measures

Define $\Omega = \{(x,y) : a < x \le y \le b\}$,
by Fubini (Tonelli),

$$\mu_F \otimes \mu_G(\Omega) = \int_{(a,b]} \int_{(a,y]} d\mu_F(x)\, d\mu_G(y)$$

$$= \int_{(a,b]} (F(y) - F(a))\, d\mu_G(y)$$

$$= \int_{(a,b]} F\, d\mu_G - F(a)(G(b) - G(a))$$

$$\mu_G([x,b]) = \lim_{n\to\infty} \mu_G((x-\tfrac{1}{n}, b])$$

$$\mu_F \otimes \mu_G(\Omega) = \int_{(a,b]} \int_{[x,b]} d\mu_G(y)\, d\mu_F(x)$$

$$= \int_{(a,b]} \lim_{n\to\infty} G(b) - G(x-\tfrac{1}{n})\, d\mu_F(x)$$

G cts ↓

$$= \int_{(a,b]} G(b) - G(x)\, d\mu_F(x)$$

$$= -\int_{(a,b]} G\, d\mu_F + G(b)(F(b) - F(a))$$

Thm (Fundamental Thm of Calc)

Given $-\infty < a < b < +\infty$, $F : [a,b] \to \mathbb{R}$,

TFAE

(a) F is abs cts on $[a,b]$

(b) $F(x) - F(a) = \int_a^x f(t)\, d\mathcal{L}(t)$ for some $f \in L^1([a,b], \mathcal{L})$.

(c) F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b], \mathcal{L})$

$F(x) - F(a) = \int_a^x F'(t)\, d\mathcal{L}(t)$.

Rmk: Given any F as in theorem, define $\tilde{F} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{F}(x) = \begin{cases} F(a) & \text{if } x < a \\ F(x) & \text{if } a \le x \le b \\ F(b) & \text{if } x > b \end{cases}$$

Then F abs cts on $[a,b]$ $\iff$ $\tilde{F}$ is abs cts on $\mathbb{R}$.

Pf: First, we show (a) => (c).

Up to subtracting a constant from F, we may assume $F(a)=0$. Take $\tilde{F}$ as in remark.

Then $\tilde{F}$ is right cts, $\tilde{F}(-\infty)=0$

Claim: $\tilde{F} \in BV$

Pf of Claim: Since $\tilde{F}$ is abs cts, for $\varepsilon=1$, choose $\delta>0$ as in defn of abs cty.

Choose $N = \lfloor \frac{b-a}{\delta} + 1 \rfloor$.

Then $\delta(N-1) \leq b-a < \delta N$

For any partition

$$a = x_0 < \dots < x_n = b,$$

the total length of all intervals in partition is $b-a$. By adding finitely many more subdivision points, we can group the intervals into N groups, where each group has total length $< \delta$.

$\{G_j\}_{j=1}^{N}$

Thus defn of abs cty gives

$$\sum_{(x_{i-1}, x_i) \in G_j} |\tilde{F}(x_i) - \tilde{F}(x_{i-1})| \leq 1$$

$$\sum_{i=1}^{n} |\tilde{F}(x_i) - \tilde{F}(x_{i-1})| \leq N$$

Thus, $T_{\tilde{F}}(b) \leq N \Rightarrow \tilde{F} \in BV$.

Thus $\mu_{\tilde{F}}$ is well-defined.

$\tilde{F}$ abs cts $\Rightarrow \mu_{\tilde{F}} \ll \mathcal{L}$

By first prop, $\tilde{F}$ diff $\mathcal{L}$-a.e.
$F' \in L^1(\mathcal{L})$,

$$\tilde{F}(x) = \int_{-\infty}^{x} \tilde{F}'(t)\, d\mathcal{L}(t).$$

Note $(c) \Rightarrow (b)$ is immediate.

Finally, to see $(b) \Rightarrow (a)$.

WLOG $f=0$ outside $[a,b]$

Then $\underbrace{\tilde{F}(x)-\tilde{F}(a)}_{\substack{\| \\ G(x)}} = \int_{-\infty}^{x} f(t)\,d\mathcal{L}(t)$ ☆

By defn, G is right cts, $G(-\infty)=0$, and BV, since

$$|G(x_i)-G(x_{i-1})| \le \int_{x_{i-1}}^{x_i} |f(t)|\,d\mathcal{L}(t)$$

and $f\in L^1(\mathbb{R},\mathcal{L})$.

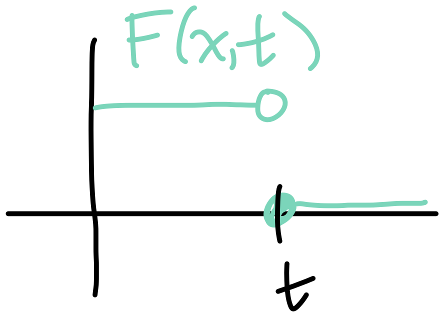
Thus μ_G is well defined and ☆ ensures $\mu_G \ll \mathcal{L}$, so G is abs cts, hence so is $\tilde{F}$. $\square$

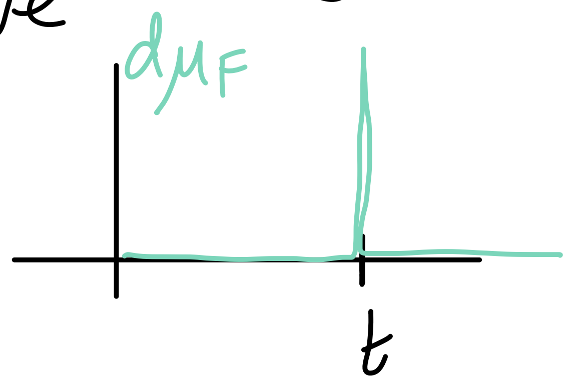
Interlude...

...why are these topics important?

Functions of Bounded Variation

- Natural regularity class for PDEs that form shocks; permits discontinuities, but still has a notion of weak derivative:





- Geometric measure theory: if $\Omega \subseteq \mathbb{R}^d$ has finite perimeter, $1_\Omega \in BV$

Absolutely Continuous Functions

• Sobolev Spaces: given $1 \le p \le +\infty$,

$$W^{1,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) : \exists\, g \in L^p(\mathbb{R}) \text{ s.t. } \int_{\mathbb{R}} u \varphi' \, d\mathcal{L} = -\int g \varphi \, d\mathcal{L} \quad \forall\, \varphi \in C^1_c(\mathbb{R}) \right.$$

Thm: $W^{1,1}(\mathbb{R})$ = abs cts fns.

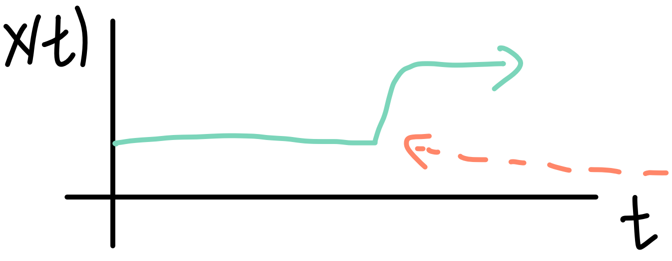
• Weak solutions of ODEs / PDEs:

STRONG

$$\begin{cases} \frac{d}{dt} x(t) = v(x(t)) \\ x(0) = x_0 \end{cases}$$

WEAK

$$x(t) = x_0 + \int_0^t v(x(s))\,ds$$



discontinuity in $x'(t)$!

Lebesgue Decomposition

- $\mu, \nu \in \mathcal{P}(X)$

$$KL(\mu|\nu) = \begin{cases} \int \log\left(\frac{d\mu}{d\nu}\right) d\mu & \text{if } \mu \ll \nu \\ +\infty & \text{otherwise} \end{cases}$$

- More generally, given $F : [0, +\infty) \to [0, +\infty)$ convex, $F'_\infty := \lim_{s \to \infty} \frac{F(s)}{s}$, we may consider the Csiszàr F-divergence:

$$\mathcal{F}(\mu|\nu) = \int_X F(f(x))\,d\nu(x) + (F'_\infty)\,\lambda(X)$$

where $d\mu = f\,d\nu + d\lambda$.

We recover KL for $F(s) = s\log s$.

- Statistical divergences of these forms arise across probability, ML, OT. For example, they fundamental in ① fast numerical methods for OT and ② generalizations of OT to measures with unequal mass.
- (Already saw importance of RN derivative in conditional expectation)

Hardy Littlewood Maximal Thm

- fundamental tool in Harmonic analysis
- used to prove well-posedness of solutions to PDEs.