# Lecture 11

260R, Advanced Measure Theory



Reminder: Presentation Topics - May 14
- May 28

**Thm:** Suppose $F, G \in BV$, $F$ right cts, $G$ cts. Then $\forall -\infty < a < b < +\infty$,

$$\int_{(a,b]} F \, d\mu_G + \int_{(a,b]} G \, d\mu_F = \underline{F(b)G(b)} \underline{- F(a)G(a)}$$

Cor: Given $F \in BV$, right cts and $G \in C_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} F(x)G'(x)\,d\mathcal{L}(x) = -\int_{\mathbb{R}} G(x)\,d\underbrace{\mu_F}_{\text{this is the weak derivative.}}(x)$$

## Radon Measures

Thm: Given a Borel measure $\mu$, for all $E \in \mathcal{B}_{\mathbb{R}^d}$,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ cpt}\}$$

Furthermore, if $\mu$ is locally finite,

$$\mu(E) = \inf\{\mu(A) : E \subseteq A,\ A \text{ open}\}$$

We proved in $d=1$ in 201a.
See section 2.4 in Maggi for $d>1$.

This theorem ensures that the behavior of $\mu$ on open/cpt sets completely determines behavior on all Borel sets.

Similarly, the behavior of $\mu$ integrated against $C_c(\mathbb{R}^d)$ completely determines behavior on all $L^1(\mu)$.

<u>Thm</u>: If $\mu$ is a locally finite Borel measure on $\mathbb{R}^d$, then $\forall u \in L^1(\mu)$, $\exists \{u_k\}_{k\in\mathbb{N}} \in C_c(\mathbb{R}^d)$ s.t. $\lim_{k\to\infty} \int_{\mathbb{R}^d} |u_k - u| d\mu = 0$.

by $\Delta$ ineq. $\Longrightarrow \lim_{k\to\infty} \int_{\mathbb{R}^d} u_k d\mu = \int_{\mathbb{R}^d} u d\mu$

We proved $d=1$ in 201a.
See section 5.1 in Maggi's book.

<u>Rmk</u>: Given two locally finite Borel measures $\mu, \nu$ on $\mathbb{R}^d$ and

$$\int u\,d\mu = \int u\,d\nu \quad \forall u \in C_c(\mathbb{R}^d),$$

then $\mu = \nu$.

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Given a locally finite Borel measure $\mu$ on $\mathbb{R}^d$, it induces a linear functional on $C_c(\mathbb{R}^d)$ via

$$L_\mu(\varphi) = \langle L_\mu, \varphi \rangle = \int \varphi \, d\mu \quad \varphi \in C_c(\mathbb{R}^d)$$

## Basic Observations

① $L$ is <u>positive</u>, $\varphi \geq 0 \Rightarrow \langle L, \varphi \rangle \geq 0$

$\Updownarrow$

② $L$ is <u>monotone</u>, $\varphi_1 \leq \varphi_2 \Rightarrow \langle L, \varphi_1 \rangle \leq \langle L, \varphi_2 \rangle$

③ $L$ is <u>bounded</u>, that is,
for any $m > 0$, $K \subseteq \mathbb{R}^d$ cpt
$$\sup \{ \langle L, \varphi \rangle : \varphi \in C_c(\mathbb{R}^d), \; |\varphi| \leq m, \; \mathrm{supp}(\varphi) \subseteq K \} < +\infty$$

$$|\varphi| = \sup_x |\varphi(x)|$$
$$\mathrm{supp}(\varphi) = \overline{\{x : \varphi(x) \neq 0\}}$$

In the case $L = L_\mu$, this is bounded by $M\mu(K) < +\infty$.

(4) $L$ is <u>continuous</u>, when $C_c(\mathbb{R}^d)$ is endowed with the following notion of convergence

$$\varphi_n \xrightarrow{C_c(\mathbb{R}^d)} \varphi \iff \varphi_n \xrightarrow{L^\infty} \varphi \text{ and } \exists \; K \text{ cpt s.t. } \operatorname{supp}\varphi_n, \operatorname{supp}\varphi \subseteq K$$

"inductive limit topology"

<u>Rmk</u>: (3) $\iff$ (4)

Sketch of (3) $\Rightarrow$ (4): Fix $\varepsilon > 0$.

Suppose $\varphi_n \to \varphi$. For $n$ suff large,

$\langle L, \varphi_n - \varphi \rangle \leq \varepsilon\mu(K)$

Thus, $\lim_{n\to\infty} \langle L, \varphi_n - \varphi \rangle \to 0$.

<u>Rmk</u>: 2 $\Rightarrow$ 3

Fix $\varphi \in C_c(\mathbb{R}^d)$, $|\varphi| \leq m$, $\text{supp}(\varphi) \subseteq K$.

There exists $\psi \in C_c(\mathbb{R}^d)$ s.t. $\psi \geq 0$ and $\psi = 1$ on $K$.
(Urysohn's Lemma)

Then $\langle L, \varphi \rangle \leq \overbrace{\langle L, m\psi \rangle}^{\in \mathbb{R}}$, so $L$ is bounded.

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The previous discussion shows that all locally finite Borel measures induce a monotone (bounded) linear functional on $C_c(\mathbb{R}^d)$.

In fact, Riesz Rep Thm ensures the converse is true.

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What if $\mu$ is a locally finite signed measure?

Then $L_\mu$ is a bounded linear functional on $C_c(\mathbb{R}^d)$, since

$$\sup\{\langle L_\mu, \varphi\rangle : \varphi \in C_c(\mathbb{R}^d),\ |\varphi| \le m,\ \mathrm{supp}(\varphi) \subseteq K\} \le m|\mu|(K)$$

Riesz Rep Thm will show that all bounded, linear functionals on $C_c(\mathbb{R}^d)$ arise from signed measures in this way.

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Lastly, Riesz Rep will give us a way to define vector valued measures...

**Def:** $\nu$ is a $\mathbb{R}^m$ vector valued locally finite Borel measure if it induced a bounded linear functional on $C_c(\mathbb{R}^d; \mathbb{R}^m)$.

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Need one more defn to state theorem...

**Def:** Given a linear functional

$$L: C_c(\mathbb{R}^d; \mathbb{R}^m) \to \mathbb{R}$$

define it total variation by

- For $A \subseteq \mathbb{R}^d$ open,
  $\|L\|(A) = \sup\{\langle L, \varphi\rangle : C_c(A; \mathbb{R}^m), |\varphi| \leq 1\}$

- For $E \subseteq \mathbb{R}^d$,
  $\|L\|(E) = \inf\{\|L\|(A) : E \subseteq A, A \text{ open}\}$

**Rmk:** If $L = L_{f d\mu}$ for $f = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}$, $f_i \in L^1(\mu)$, $\mu$ locally finite Borel measure, then for $\varphi \in C_c(\mathbb{R}^d; \mathbb{R}^m)$,

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^d} \varphi \cdot f \, d\mu$$

When $L$ is of this form,

$$|L|(A) = \sup\left\{ \int \varphi \cdot f \, d\mu : \text{~~~} \right\}$$

in fact, "=" since $C_c(\mathbb{R}^d)$ dense in $L^1(\mu)$

$$\leq \int_A |f| \, d\mu$$

So $|L|$ coincides with $|f| d\mu$ on open sets; by outer regularity, it coincides for all Borel sets.

$$|L| = |f| d\mu = \underbrace{|f d\mu|}_{\text{total variation of signed measure}}.$$

<u>Thm</u> (Riesz Representation): Given a bounded linear functional $L: C_c(\mathbb{R}^d; \mathbb{R}^m)$,

(i) $|L|$ is a locally finite Borel measure on $\mathbb{R}^d$

(ii) there exists $g: \mathbb{R}^d \to \mathbb{R}^m$ measurable with $|g| = 1$ $|L|$-a.e. and

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^d} \varphi \cdot g \, d|L|$$

for all $\varphi \in C_c(\mathbb{R}^d; \mathbb{R}^m)$.

(Note that $g \in L^1_{loc}(|L|)$.)

In the case $m=1$, we see that any bdd, linear functional may be expressed as

$$\langle L, \varphi \rangle = \int \varphi \underbrace{g \, d|L|}_{\nu :=}$$

this is a locally finite signed measure

$$\nu^+ = \mathbb{1}_{\{g=1\}} d|L|$$

$$\nu^- = \mathbb{1}_{\{g=-1\}} d|L|$$