

Lecture 12

260R, Advanced Measure Theory
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Reminder: Presentation Topics - May 14
- May 28

Thm: If μ is a locally finite Borel measure on \mathbb{R}^d , then
 $\forall u \in L^1(\mu), \exists \{u_k\}_{k \in \mathbb{N}} \in C_c(\mathbb{R}^d)$
s.t. $\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} |u_k - u| d\mu = 0$.

Lemma: Given two locally finite Borel measures μ, ν on \mathbb{R}^d and
 $\int u d\mu = \int u d\nu \quad \forall u \in C_c(\mathbb{R}^d)$,
then $\mu = \nu$.

In fact, we will prove something stronger:

Lemma: Given a locally finite signed Borel measure σ on \mathbb{R}^d , if $\int \varphi d\sigma^+ \geq \int \varphi d\sigma^-$ for $\varphi \in C_c(\mathbb{R}^d)$, $\varphi \geq 0 \Rightarrow \int \varphi d\sigma \geq 0$ then σ is a positive measure.

We see that the first lemma follows from the second by taking $\sigma = \pm(\mu - \nu)$.

Our proof of second lemma relies on...
↓ Folland 4.15

Lemma (Urysohn): Given $A, B \subseteq \mathbb{R}^d$ disjoint, closed, there exists $f \in C(\mathbb{R}^d; [0, 1])$ s.t. $f = 0$ on A and $f = 1$ on B .

Pf of Second Lemma:

Consider the Jordan-Hahn decomposition $\sigma = \sigma^+ - \sigma^-$, $\mathbb{R}^d = P \sqcup N$
concentrated on P \uparrow \uparrow N

By inner regularity of σ^- , it suffices to show $\sigma^-(K) = 0$ for any $K \subset \text{pt}$, $K \subseteq N$.

Choose $R > 0$ suff large so $K \subseteq B_R(0)$.

Fix E open, $E \supseteq N$, by Urysohn, $\exists \varphi \in C(\mathbb{R}^d; [0, 1])$ s.t. $0 \leq \varphi \leq 1$
 $\varphi = 0$ on $(E \cap B_R)^c$
 $\varphi = 1$ on $E \cap B_R(0)$

$$\sigma^-(K) \leq \int \varphi d\sigma^- \leq \int \varphi d\sigma^+ \leq \sigma^+(E \cap B_R(0))$$

cty from below \downarrow

$$\sigma^-(K) \leq \sigma^+(E)$$

outer regularity \downarrow

$$\sigma^-(K) \leq \sigma^+(N) = 0$$

Given a locally finite Borel measure μ on \mathbb{R}^d , it induces a linear functional on $C_c(\mathbb{R}^d)$ via

$$L(\varphi) = \langle L, \varphi \rangle = \int \varphi d\mu \quad \varphi \in C_c(\mathbb{R}^d)$$

Basic Observations

- ① L is positive, $\varphi \geq 0 \Rightarrow \langle L, \varphi \rangle \geq 0$
 $\varphi(x) \geq 0 \forall x$
- ② L is monotone, $\varphi_1 \leq \varphi_2 \Rightarrow \langle L, \varphi_1 \rangle \leq \langle L, \varphi_2 \rangle$
 $\varphi_1(x) \leq \varphi_2(x) \forall x$
- ③ L is bounded, that is,
for any $m > 0$, $K \subseteq \mathbb{R}^d$ cpt
 $\sup \{ \langle L, \varphi \rangle : \varphi \in C_c(\mathbb{R}^d), |\varphi(x)| \leq m \forall x, |\varphi| \leq m, \text{supp}(\varphi) \subseteq K \} < +\infty$

$$\text{supp}(\varphi) = \overline{\{x: \varphi(x) \neq 0\}}$$

④ L is continuous, when $C_c(\mathbb{R}^d)$ is endowed with the following notion of convergence

$$\varphi_n \xrightarrow{C_c(\mathbb{R}^d)} \varphi \iff \varphi_n \xrightarrow{L^\infty} \varphi \text{ and } \exists K \text{ cpt s.t.}$$

$$\text{supp } \varphi_n, \text{supp } \varphi \subseteq K$$

"inductive limit topology"

Rmk: (3) \Leftrightarrow (4)

Rmk: (2) \Rightarrow (3)

The previous discussion shows that all locally finite Borel measures induce a monotone (bounded) linear functional on $C_c(\mathbb{R}^d)$.

In fact, Riesz Rep Thm ensures the converse is true.

What if μ is a locally finite signed measure?

Then L_μ is a bounded linear functional on $C_c(\mathbb{R}^d)$.

Riesz Rep Thm will show that all bounded, linear functionals on $C_c(\mathbb{R}^d)$ arise from signed measures in this way.

Def: Given a linear functional

$$L: C_c(\mathbb{R}^d; \mathbb{R}^m) \rightarrow \mathbb{R}$$

define its total variation by

• For $A \subseteq \mathbb{R}^d$ open,

$$|L|(A) = \sup \{ \langle L, \varphi \rangle : \varphi \in C_c(A; \mathbb{R}^m), |\varphi| \leq 1 \}$$

• For $E \subseteq \mathbb{R}^d$

$$|L|(E) = \inf \{ |L|(A) : E \subseteq A, A \text{ open} \}$$

Rmk: If $L = L_{f d\mu}$ for $f = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}$,

$f_i \in L^1(\mu)$, μ locally finite Borel measure, then for

$$\varphi \in C_c(\mathbb{R}^d; \mathbb{R}^m),$$

$$\langle L_{f d\mu}, \varphi \rangle = \int_{\mathbb{R}^d} \varphi \cdot f d\mu$$

When L is of this form,

$$|L| = |f|d\mu = |f|d\nu.$$

total variation
of signed
measure, $m=1$

Lastly, Riesz Rep will give us a way to define vector valued measures...

Def: ν is a \mathbb{R}^m vector valued locally finite Borel measure if it induces a bounded linear functional on $C_c(\mathbb{R}^d, \mathbb{R}^m)$.

Thm (Riesz Representation):

Given a bounded linear functional $L: C_c(\mathbb{R}^d; \mathbb{R}^m)$,

(i) $|L|$ is a locally finite Borel measure on \mathbb{R}^d

(ii) there exists $g: \mathbb{R}^d \rightarrow \mathbb{R}^m$ measurable with $|g| = 1$ $|L|$ -a.e. and

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^d} \varphi \cdot g d|L|$$

for all $\varphi \in C_c(\mathbb{R}^d; \mathbb{R}^m)$.

(Note that $g \in L^1_{loc}(|L|)$.)

In the case $m=1$, we see that any bdd, linear functional may be expressed as

$$\langle L, \varphi \rangle = \int \varphi g d|L|$$

$\nu := \int$

this is a locally finite signed measure

$$\nu^+ = \int_{\{g=1\}} d|L|$$

$$\nu^- = \int_{\{g=-1\}} d|L|$$

Furthermore, if L is monotone, that is

$$\varphi \in C_c(\mathbb{R}^d), \varphi \geq 0 \Rightarrow \int \varphi \cdot g d|L| \geq 0$$

by lemma, $g d|L|$ is a positive measure, so $g=1$ $|L|$ -a.e.

Thus, we see that, when $m=1$ and L is monotone, L is induced by the locally finite Borel measure $|L|$. \square

Our proof of the theorem will rely on some well-known results from previous courses...

Recall:

✓ Folland 4.41

Prop (Partition of Unity): Given $K \subseteq \mathbb{R}^d$ cpt and $\{U_j\}_{j=1}^n$ an open cover of K , there exists $\{h_j\}_{j=1}^n \subseteq C(\mathbb{R}^d; [0, 1])$ such that

(i) $\text{supp } h_j \subseteq U_j \quad \forall j$

(ii) $\sum_{j=1}^n h_j(x) = 1 \quad \forall x \in K$

✓ Folland 6.15

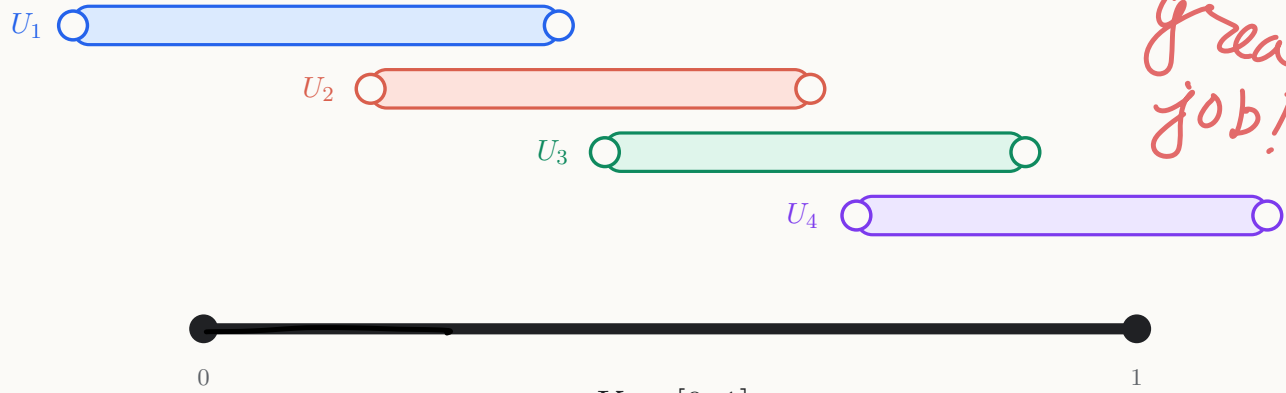
Thm: Given μ a σ -finite Borel measure on \mathbb{R}^d ,
 $(L^1(\mu))^* = L^\infty(\mu)$;

that is, for all $\phi \in (L^1(\mu))^*$, $\exists!$
 $g \in L^\infty(\mu)$ s.t. $\langle \phi, f \rangle = \int f g d\mu \quad \forall f \in L^1(\mu)$.
and $\|\phi\|_{(L^1(\mu))^*} = \|g\|_{L^\infty(\mu)}$.

Partition of unity subordinate to an open cover

A compact set K is covered by open sets U_j ; the functions ϕ_j live inside those sets and add to 1 on K .

1. Open cover of a compact set



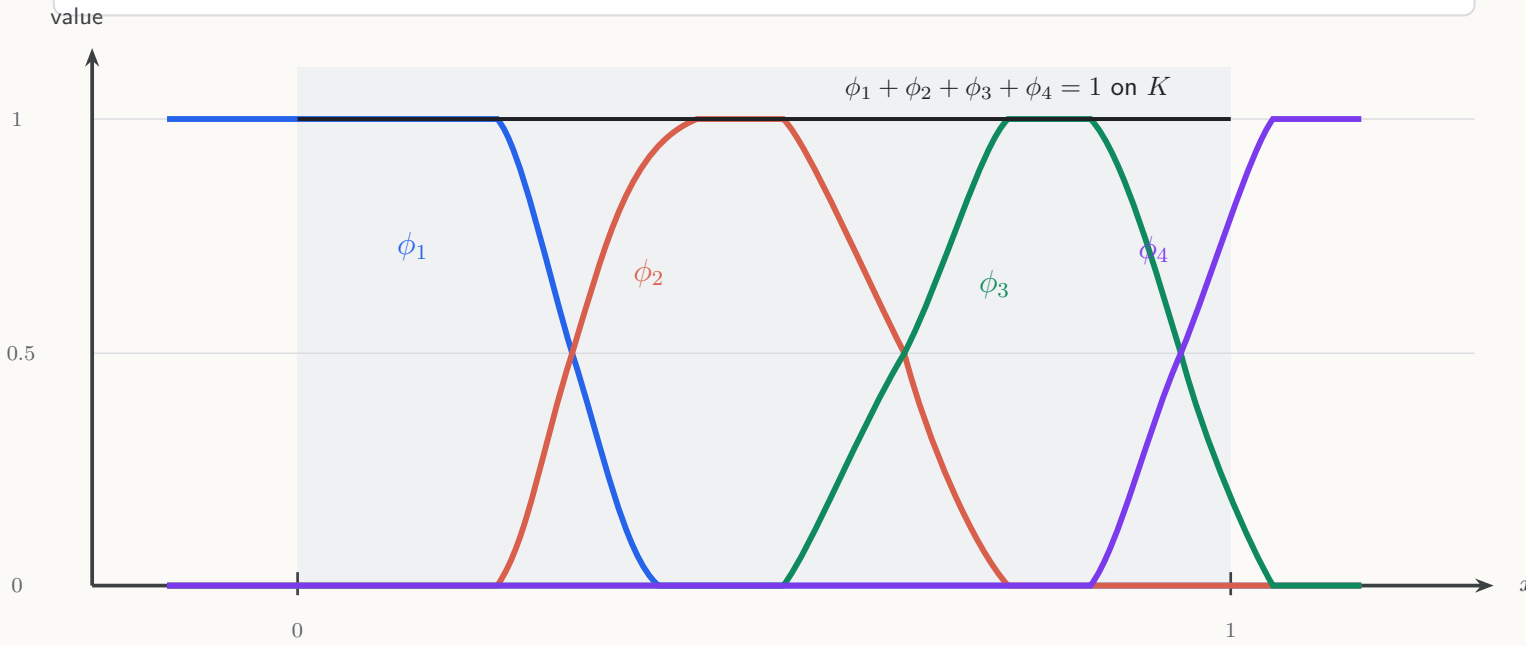
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job!

Each point of K lies in at least one open set in the cover. $K = [0, 1]$

2. Continuous bump functions forming the partition

Subordinate: $\text{supp } \phi_j \subset U_j$

Partition on K : $\phi_j \geq 0, \sum_j \phi_j = 1$



$\text{supp } \phi_1 \subset U_1$

$\text{supp } \phi_2 \subset U_2$

$\text{supp } \phi_3 \subset U_3$

$\text{supp } \phi_4 \subset U_4$

Intervals are used for clarity; for a general compact K , the same idea is local bump functions adapted to the open cover.

Pf

First, we show (i), that $|L|$ is a locally finite Borel measure.

Step (1): $|L|$ is an outer measure

Clear that $|L|(\emptyset) = 0$. We now show $E \subseteq \bigcup_{n=1}^{\infty} E_n$ then

$$|L|(E) \leq \sum_{n=1}^{\infty} |L|(E_n). \quad \odot$$

CLAIM: If $\{A_n\}_{n=1}^{\infty}$ open, then $|L|(\bigcup_n A_n) \leq \sum_{n=1}^{\infty} |L|(A_n)$.

Suppose the claim holds.

Fix $\varepsilon > 0$. WLOG, $|L|(E_n) < +\infty$

$\forall n \in \mathbb{N}$. $\exists \{A_n\}_{n \in \mathbb{N}}$ open s.t.

$$|L|(A_n) \leq |L|(E_n) + \frac{\varepsilon}{2^n}$$

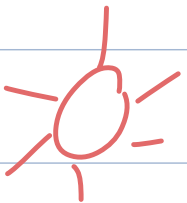
and $E_n \subseteq A_n$.

Thus,

$$|L|(E) \stackrel{\text{CLAIM}}{=} |L|\left(\bigcup_n A_n\right) = \sum_n |L|(A_n)$$

$$\leq \sum_n |L|(E_n) + \frac{\varepsilon}{2^n}$$

$$\leq \left(\sum_n |L|(E_n)\right) + \varepsilon$$

This shows .

It remains to prove the claim.

Let $A = \bigcup_{n=1}^{\infty} A_n$. Fix $\varphi \in C_c(A; \mathbb{R}^m)$
 $|\varphi| \leq 1$. Then $\text{supp } \varphi \subseteq \bigcup_{n=1}^{\infty} A_n$

and \exists finite subcover,
 $\text{supp } \varphi \subseteq \bigcup_{n=1}^N A_n$.

Let $\{\varphi_n\}_{n=1}^{\infty} \subseteq C_c(\mathbb{R}^d; [0, 1])$ be
a partition of unity satisfying
 $\text{supp } \varphi_n \subseteq A_n$ and $\sum_{n=1}^{\infty} \varphi_n = 1$ on

supp φ .

$$\text{Then } \varphi = \sum_{n=1}^{\infty} \varphi_n \varphi$$
$$\langle L, \varphi \rangle = \sum_{n=1}^{\infty} \langle L, \varphi_n \varphi \rangle \leq \sum_{n=1}^{\infty} |L|(A_n)$$

$\varphi_n \varphi \in C_c(A_n; \mathbb{R}^m)$
 $|\varphi_n \varphi| \leq 1$

$$\text{Therefore } |L|(A) \leq \sum_{n=1}^{\infty} |L|(A_n).$$

Step (2): $|L|$ is a Borel measure

Recall from Lecture 1, it suffices to show that $\text{dist}(E_1, E_2) > 0$ implies $|L|(E_1 \cup E_2) \geq |L|(E_1) + |L|(E_2)$.

If E_1 and E_2 are open and $\text{dist}(E_1, E_2) > 0$, then for any $\varphi_1 \in C_c(E_1; \mathbb{R}^m)$, $\varphi_2 \in C_c(E_2; \mathbb{R}^m)$, $\varphi_1 + \varphi_2 \in C_c(E_1 \cup E_2; \mathbb{R}^{2m})$.

Then

$$|L|(E_1 \cup E_2) \geq \langle L, \varphi_1 + \varphi_2 \rangle \\ = \langle L, \varphi_1 \rangle + \langle L, \varphi_2 \rangle$$

Thus, $|L|(E_1 \cup E_2) \geq |L|(E_1) + |L|(E_2)$

More generally, for any $E_1, E_2 \subseteq \mathbb{R}^d$,

$$0 < \text{dist}(E_1, E_2) = \text{dist}(\bar{E}_1, \bar{E}_2),$$

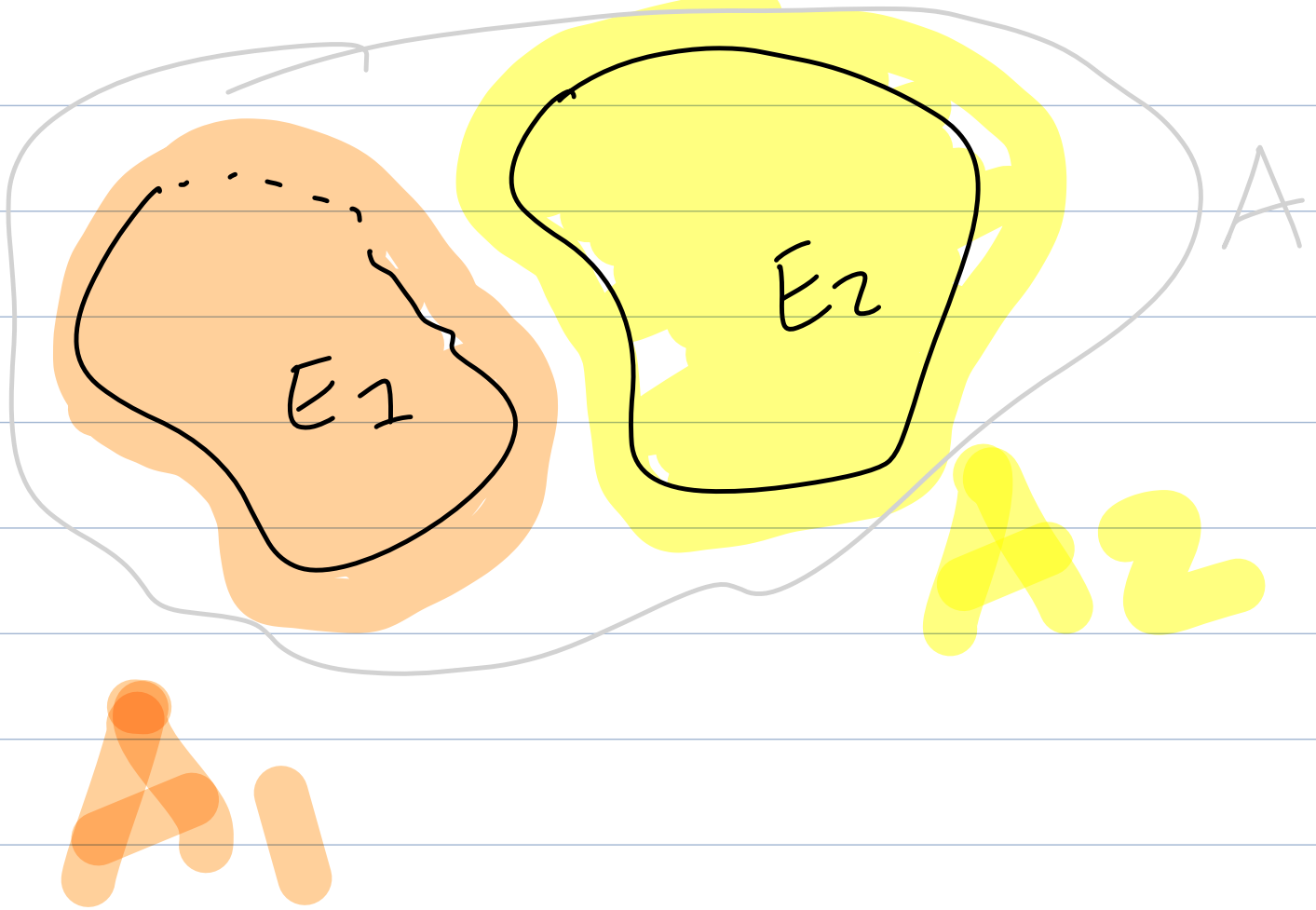
so $\exists A_j$ open, $A_j \supseteq \bar{E}_j$ so that $0 < \text{dist}(A_1, A_2)$.

take $A_j = \{x \in \mathbb{R}^d : \text{dist}(x, \bar{E}_j) < r\}$
for r suff small.

For any A open, $A \supseteq E_1 \cup E_2$,
 $\text{dist}(A \cap A_1, A \cap A_2) \geq \text{dist}(A_1, A_2) > 0$,

$E_j \subseteq A_j \cap A$, \leftarrow monotonicity of outer measure

$$|L|(A) \geq |L|((A \cap A_1) \cup (A \cap A_2)) \\ \geq |L|(A \cap A_1) + |L|(A \cap A_2)$$



Thus

$$|L|(E_1 \cup E_2) \geq |L|(A \cap A_1) + |L|(A \cap A_2)$$

monotonicity \downarrow

$$\geq |L|(E_1) + |L|(E_2)$$

$E_j \subseteq A \cap A_j$

Thus $|L|$ is a Borel measure. Since L is bounded, for any open ball B , $|L|(B) < +\infty$. Thus L is locally finite. \square