# Lecture 13

260R, Advanced Measure Theory


Reminder:

No Class on Tuesday, May 26th
Final Presentations on Thursday, May 28th

Def: Given a linear functional $L : C_c(\mathbb{R}^d; \mathbb{R}^m) \to \mathbb{R}$ define it total variation by

- For $A \subseteq \mathbb{R}^d$ open,
  $|L|(A) = \sup\{\langle L, \varphi\rangle : C_c(A; \mathbb{R}^m), |\varphi| \leq 1\}$
- For $E \subseteq \mathbb{R}^d$,
  $|L|(E) = \inf\{|L|(A) : E \subseteq A, A \text{ open}\}$

**Thm** (Riesz Representation): Given a bounded linear functional $L : C_c(\mathbb{R}^d; \mathbb{R}^m)$,

(i) $|L|$ is a locally finite positive Borel measure on $\mathbb{R}^d$

(ii) there exists $g : \mathbb{R}^d \to \mathbb{R}^m$ measureable with $|g| = 1$ $|L|$-a.e. and

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^d} \varphi \cdot g \, d|L|$$

for all $\varphi \in C_c(\mathbb{R}^d; \mathbb{R}^m)$.

Recall:

Folland 4.41

**Prop** (Partition of Unity): Given $K \subseteq \mathbb{R}^d$ cpt and $\{U_j\}_{j=1}^n$ an open cover of $K$, there exists $\{h_j\}_{j=1}^n \subseteq C(\mathbb{R}^d; [0,1])$ such that

(i) $\operatorname{supp} h_j \subseteq U_j \quad \forall j$

(ii) $\sum_{j=1}^n h_j(x) = 1 \quad \forall x \in K$

Folland 6.15

**Thm**: Given $\mu$ a $\sigma$-finite Borel measure $\mu$ on $\mathbb{R}^d$,

$$(L^1(\mu))^* = L^\infty(\mu);$$

that is, for all $\phi \in (L^1(\mu))^*$, $\exists!$ $g \in L^\infty(\mu)$ s.t. $\langle \phi, f \rangle = \int f g \, d\mu \ \forall f \in L^1(\mu)$. and $\|\phi\|_{(L^1(\mu))^*} = \|g\|_{L^\infty(\mu)}$.

Rmk:

- $C_c(\mathbb{R}^d;\mathbb{R}^m)$ with the topology $\varphi_n \to \varphi \iff \exists K$ cpt s.t. $\operatorname{supp}\varphi_n, \operatorname{supp}\varphi \subseteq K$ and $\varphi_n \to \varphi$ uniformly.

  This topology is not metrizable. Instead, $C_c(\mathbb{R}^d;\mathbb{R}^m)$ is a topological vector space.

- We can still define $(C_c(\mathbb{R}^d;\mathbb{R}^m))^* = \{$bdd linear functionals on $C_c(\mathbb{R}^d;\mathbb{R}^m)\}$

  Riesz Representation

  $= \{ g\,d\mu :$ $\mu$ loc finite pos Borel, $|g|=1$ $\mu$-a.e.$\}$

We defined this to be the space of vector valued measures.

Pf:
Last time, we showed (i), that $|L|$ is a locally finite positive Borel measure.

Now, we show (ii).

Strategy...

Define $M : C_c(\mathbb{R}^d, [0,+\infty)) \to [0,+\infty]$,

$$\langle M, \varphi \rangle = \sup \{ \langle L, \psi \rangle : C_c(\mathbb{R}^d; \mathbb{R}^m), \ |\psi(x)| \le \varphi(x) \ \forall x \}$$

① $M$ is additive, positively homogeneous deg 1, monotone

② $\langle M, \varphi \rangle \le \int \varphi \, d|L|$

③ $(L^1(|L|))^* = L^\infty(|L|)$.

## Step 1

**monotone:** $\varphi_1 \le \varphi_2 \Rightarrow \langle m, \varphi_1 \rangle \le \langle m, \varphi_2 \rangle$

This follows, since constraint set on RHS contains constraint set on LHS

**homogeneity:** $c \geq 0 \Rightarrow \langle m, c\varphi \rangle = c \langle m, \varphi \rangle$

$|\psi| \le c\varphi \Leftrightarrow |\frac{1}{c}\psi| \le \varphi$

**additivity:** $\langle m, \varphi_1 + \varphi_2 \rangle = \langle m, \varphi_1 \rangle + \langle m, \varphi_2 \rangle$

"$\geq$": $|\psi_1| \le \varphi_1, \ |\psi_2| \le \varphi_2$

$\Rightarrow |\psi_1 + \psi_2| \le \varphi_1 + \varphi_2$

Finally, to see "$\leq$," suppose $\psi \in C_c(\mathbb{R}^d;\mathbb{R}^m)$, $|\psi| \leq \varphi_1 + \varphi_2$.

Define $\psi_i(x) = \begin{cases} \dfrac{\varphi_i(x)\cdot\psi(x)}{\varphi_1(x)+\varphi_2(x)} & \text{if } \varphi_1(x)+\varphi_2(x)>0 \\ 0 & \text{otherwise} \end{cases}$

Then $\psi_i \in C_c(\mathbb{R}^d;\mathbb{R}^m)$, $|\psi_i(x)| \leq \varphi_i(x)$, and $\psi_1+\psi_2=\psi$, so

$$\langle L,\psi\rangle = \langle L,\psi_1\rangle + \langle L,\psi_2\rangle \leq \langle m,\varphi_1\rangle + \langle m,\varphi_2\rangle.$$

$$\Rightarrow \langle m,\varphi_1+\varphi_2\rangle \leq \langle m,\varphi_1\rangle + \langle m,\varphi_2\rangle.$$

Step 2: WTS $\langle m, \varphi \rangle \leq \int \varphi \, d|L|$.

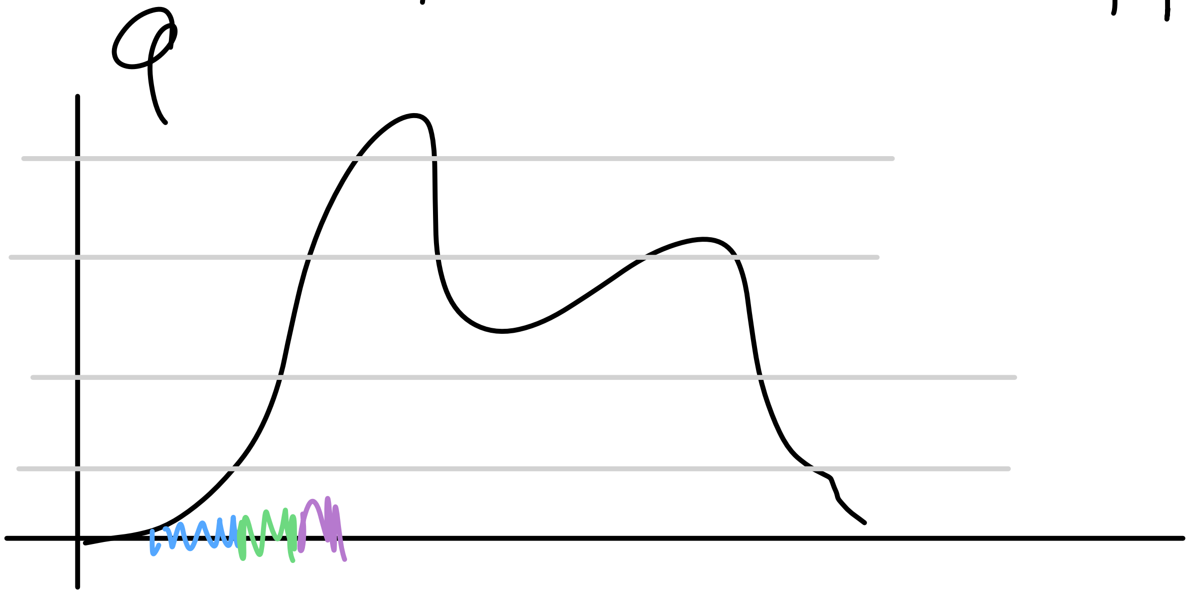
Fix $\varphi \in C_c(\mathbb{R}^d; [0, +\infty))$, $\varepsilon > 0$.

Choose $\{t_n\}_{n=0}^{N} \subseteq \mathbb{R}$ s.t.

$$t_0 < 0 < t_1 < \dots < t_{N-1} < \|\varphi\|_\infty < t_N$$
$$t_{N+1} - t_N < \varepsilon$$

$$E_n = \{x \in \operatorname{supp} \varphi : t_{n-1} < \varphi(x) \leq t_n\}$$

$\{E_n\}_{n=1}^{N}$ partitions $\operatorname{supp} \varphi$



By outer regularity of $|L|$, $\exists$ $\tilde{A}_n$ s.t. $E_n \subseteq \tilde{A}_n$ and

☼ $\quad |L|(\tilde{A}_n) \le |L|(E_n) + \frac{\varepsilon}{N} \quad \forall n.$

The set $A_n := \{x \in \tilde{A}_n : \varphi(x) < t_n + \varepsilon\}$ also satisfies ☼.

Let $\zeta_n$ be a partition of unity subordinated to open cover $\{A_n\}_{n=1}^N$ of $\operatorname{supp}\varphi$.

Then $\varphi = \sum_{n=1}^N \zeta_n \varphi$, so by Step 1,

$$\langle m, \varphi \rangle = \sum_{n=1}^N \langle m, \overbrace{\zeta_n \varphi}^{\le \zeta_n(t_n+\varepsilon)} \rangle$$

$$\le \sum_{n=1}^N \langle m, (t_n+\varepsilon)\zeta_n \rangle$$

$$= \sum_{n=1}^N (t_n+\varepsilon)\langle m, \zeta_n \rangle.$$

Note that

$$\begin{aligned}
\langle m, \zeta_n \rangle &= \sup\{\langle L, \psi\rangle : C_c(\mathbb{R}^d; \mathbb{R}^m),\ |\psi(x)| \le \zeta_n(x)\ \forall x\} \\
&\le \sup\{\langle L, \psi\rangle : C_c(A_n; \mathbb{R}^m),\ |\psi(x)| \le 1\ \forall x\} \\
&= |L|(A_n) \\
&\le |L|(E_n) + \frac{\varepsilon}{N}
\end{aligned}$$

Combining these,

$$\begin{aligned}
\langle m, \varphi \rangle &\le \sum_{n=1}^{N} \underbrace{(t_n + \varepsilon)}_{\le t_{n-1} + 2\varepsilon} \langle m, \zeta_n \rangle \\
&\le \sum_{n=1}^{N} (t_{n-1} + 2\varepsilon)\left(|L|(E_n) + \frac{\varepsilon}{N}\right) \\
&\le \sum_{n=1}^{N} \int_{E_n} \varphi \, d|L| + \frac{\overbrace{t_{N-1}}^{\le \|\varphi\|_\infty} \varepsilon}{N} + 2\varepsilon |L|(E_N) + \frac{2\varepsilon^2}{N} \\
&= \int_{\mathbb{R}^d} \varphi \, d|L| + \|\varphi\|_\infty \varepsilon + 2\varepsilon |L|(\operatorname{supp} \varphi) + 2\varepsilon^2
\end{aligned}$$

Sending $\varepsilon \to 0$ gives the result.

<u>Step 3</u>: For $e \in S^{m-1} \subseteq \mathbb{R}^m$, define

$L_e : C_c(\mathbb{R}^d) \to \mathbb{R}$ by

$$\langle L_e, \varphi \rangle = \langle L, \varphi e \rangle$$

By Step 2, for any $\varphi \in C_c(\mathbb{R}^d)$,

$$\begin{aligned}\langle L_e, \varphi \rangle &\leq \sup\{\langle L, \psi \rangle : \psi_c(\mathbb{R}^d; \mathbb{R}^m), |\psi(x)| \leq |\varphi(x)| \; \forall x\} \\ &= \langle m, |\varphi| \rangle \\ &\leq \int |\varphi| \, d|L|\end{aligned}$$

By density of $C_c(\mathbb{R}^d)$ in $L^1(|L|)$, we may extend $L_e$ as a linear functional on $L^1(|L|)$.

Since $L_e \in (L^1(|L|))^*$, $\exists$ $g_e \in L^\infty(|L|)$ s.t., $\forall u \in L^1(|L|)$,

$$\langle L_e, u\rangle = \int u g_e \, d|L| .$$

Define $g : \mathbb{R}^d \to \mathbb{R}^m$ by $g^{(i)} = g_{e_i}$.

$g \in L^\infty(\mathbb{R}^d; \mathbb{R}^m)$

This gives, $\forall \varphi \in C_c(\mathbb{R}^d; \mathbb{R}^m)$,

$$\langle L, \varphi\rangle = \langle L, \sum_{i=1}^m \varphi^{(i)} e_i\rangle$$

$$= \sum_{i=1}^m \int \varphi^{(i)} g^{(i)} \, d|L|$$

$$= \int \varphi \cdot g \, d|L| .$$

It remains to show
$|g(x)| = 1$ for $|L|$-a.e. $x$.

Next time ☺