Lecture 14

260R, Advanced Measure Theory

Reminder:

No Class on Tuesday, May 26th

Final Presentations on Thursday, May 28th

Def: Given a linear functional

$$L : C_c(\mathbb{R}^d; \mathbb{R}^m) \to \mathbb{R}$$

define it total variation by

- For $A \subseteq \mathbb{R}^d$ open,

  $|L|(A) = \sup\{ \langle L, \varphi \rangle : C_c(A; \mathbb{R}^m), |\varphi| \leq 1 \}$

- For $E \subseteq \mathbb{R}^d$,

  $|L|(E) = \inf\{ |L|(A) : E \subseteq A, A \text{ open} \}$

__Thm__ (Riesz Representation): Given a bounded linear functional $L : C_c(\mathbb{R}^d;\mathbb{R}^m)$,

(i) $|L|$ is a locally finite positive Borel measure on $\mathbb{R}^d$

(ii) there exists $g : \mathbb{R}^d \to \mathbb{R}^m$ measurable with $|g| = 1$ $|L|$-a.e. and

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^d} \varphi \cdot g \, d|L|$$

for all $\varphi \in C_c(\mathbb{R}^d;\mathbb{R}^m)$.

Last time, we showed $\exists\ g\in L^\infty(\mathbb{R}^d;\mathbb{R}^m)$ s.t. $\forall\ \varphi\in C_c(\mathbb{R}^d;\mathbb{R}^m)$,

$$\langle L,\varphi\rangle = \int \varphi\cdot g\,d|L|.$$

It remains to show $|g|=1$ $|L|$-a.e.

Note that

$$|L|(A)=\sup\{\overbrace{\langle L,\varphi\rangle}^{\int \varphi\cdot g\,d|L|} : \varphi\in C_c(A;\mathbb{R}^m), |\varphi|\le$$

$$\le \int_A |g|\,d|L|$$

Thus, $E := \{x : |g(x)|=0\}$, so $|L|(E)=0$ hence $|g|>0$ $|L|$-a.e.

For any A open and bounded, $|L|$ locally finite ensures

$$\frac{g}{|g|} \in L^1(A, |L|; \mathbb{R}^m)$$

By density, $\exists\, \varphi_n \in C_c(A; \mathbb{R}^m)$
s.t. $\varphi_n \xrightarrow{L} \frac{g}{|g|}$ in $L^1(A, |L|; \mathbb{R}^m)$

Define

$$\tilde{\varphi}_n(x) := \begin{cases} \pi_{S^{m-1}}(\varphi_n(x)) & \text{if } |\varphi_n(x)| \geq 1 \\ \varphi_n(x) & \text{otherwise.} \end{cases}$$

Then $\tilde{\varphi}_n \in C_c(A; \mathbb{R}^m)$ and

$$\int_A \left|\tilde{\varphi}_n - \frac{g}{|g|}\right| d|L| \leq \int_A \left|\varphi_n - \frac{g}{|g|}\right| d|L| \xrightarrow{n\to\infty} 0$$

Finally,

$$\int_A \overbrace{\left|\tilde{\varphi}_n \cdot g - |g|\right|}^{\left|\left(\tilde{\varphi}_n - \frac{g}{|g|}\right)\cdot g\right|} d|L| \leq \|g\|_\infty \int_A \left|\tilde{\varphi}_n - \frac{g}{|g|}\right| \xrightarrow{n\to\infty} 0$$

Thus,

$$|L|(A) \geq \int_A \tilde{\varphi}_n \cdot g \, d|L|$$

$$\downarrow n \to \infty$$

$$|L|(A) \geq \int_A |g| \, d|L| \geq |L|(A)$$

$$\Rightarrow |g| \, d|L| = |L| \Rightarrow |g| = 1 \ |L| \text{ a.e.} \quad \square$$

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aka weak-*

μ_i signed

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix}$$

Vague-convergence

$\mathbb{R}^m$ valued Borel measures

Def: Given locally finite v.v measures $\{\mu_n\}_{n\in\mathbb{N}}, \mu$, we say $\mu_n \to \mu$ vaguely if

$$\lim_{n\to\infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu \quad \forall \varphi \in C_c(\mathbb{R}^d; \mathbb{R}^m)$$

Ex:

- If $x_n \to x$ in $\mathbb{R}^d$, $\delta_{x_n} \xrightarrow{n\to\infty} \delta_x$ vaguely
- $\mu_n = n^d \mathbb{1}_{[0,\frac{1}{n}]} \, d\mathcal{L}^d \xrightarrow{n\to\infty} \delta_0$

Thm (Portmanteau) Given locally finite positive Borel measures $\{\mu_n\}_{n\in\mathbb{N}}, \mu$

TFAE

(i) $\mu_n \to \mu$ vaguely

(ii) $\forall$ K cpt and A open in $\mathbb{R}^d$

$$\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$$

$$\limsup_{n\to\infty} \mu_n(K) \leq \mu(K)$$

(iii) $\forall$ $E \in \mathcal{B}_{\mathbb{R}^d}$ bounded with $\mu(\partial E) = 0$, $\lim_{n\to\infty} \mu_n(E) = \mu(E)$.

Vague compactness

Thm: Given locally finite positive Borel measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}^d$ s.t. $\forall K$ cpt

$$\sup_{n\in\mathbb{N}} \mu_n(K) < +\infty,$$

then $\exists\ \mu$ locally finite Borel measure and n_k s.t.
$\mu_{n_k} \Rightarrow \mu$ vaguely.

Fact: $C_c(\mathbb{R}^d)$ endowed with inductive limit topology is separable.

Pf: Let $\mathcal{F} \subseteq C_c(\mathbb{R}^d)$ be a countable dense subset. For any $\varphi \in C_c(\mathbb{R}^d)$

$$\sup_n \left|\int \varphi \, d\mu_n\right| \le \|\varphi\|_\infty \sup_n \mu_n(\operatorname{supp}\varphi) < +\infty$$

Thus, $\forall \varphi \in \mathcal{F}$, $\exists L\varphi$ s.t. $\int \varphi \, d\mu_{n_k} \to L\varphi$. By a diagonal argument, we may assume the same subsequence n_k works for all $\varphi \in \mathcal{F}$!

Note that $\varphi \mapsto L\varphi$ is linear on $\mathcal{F}$ and $\varphi \ge 0 \Rightarrow L\varphi \ge 0$.

Suppose $\{\varphi_n\}_{n\in\mathbb{N}}, \{\psi_m\}_{n\in m} \subseteq \mathcal{F}$ satisfy $\varphi_n \to \varphi$ & $\psi_m \to \varphi_p$

then

$$\lim_{n,m\to\infty} |L\varphi_n - L\psi_m|$$

$$= \lim_{n,m,k\to\infty} \left|\int \varphi_n \, d\mu_{n_k} - \int \psi_m \, d\mu_{n_k}\right|$$

$$\leq \lim_{n,m\to\infty} \|\varphi_n - \psi_m\|_\infty \overbrace{\sup_k \mu_{n_k}(K)}^{<+\infty}$$

thus, we can define $L : C_c(\mathbb{R}^d) \to \mathbb{R}$,

$$\langle L, \varphi\rangle = \lim_{n\to\infty} L\varphi_n$$

where $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\varphi_n \to \varphi$

L is a monotone, linear functional on $C_c(\mathbb{R}^d)$, so it's bounded. By Riesz Representation, $\exists$ locally finite positive

Borel measure μ s.t.

$\forall\, \varphi \in C_c(\mathbb{R}^d)$

$$\langle L, \varphi \rangle = \int \varphi \, d\mu .$$

For $\varphi \in \mathcal{F}$,

$$\lim_{k\to\infty} \int \varphi \, d\mu_{n_k} = L\varphi = \langle L, \varphi \rangle = \int \varphi \, d\mu .$$

Thus, $\forall\, \psi \in C_c(\mathbb{R}^d)$,

$\varphi, \psi \subseteq K$

$$\left| \int \psi \, d\mu_{n_k} - \int \psi \, d\mu \right|$$

$$\leq \overbrace{\overset{\text{(I)}}{\left| \int \varphi \, d\mu_{n_k} - \int \psi \, d\mu_{n_k} \right|}}^{\leq \|\varphi - \psi\|_\infty \sup \mu_{n_k}(K)} + \overset{\text{(II)}}{\left| \int \varphi \, d\mu - \int \psi \, d\mu \right|}$$

$$+ \overset{\text{(III)}}{\left| \int \varphi \, d\mu_{n_k} - \int \varphi \, d\mu \right|}$$

Since $\mathcal{F}$ is dense in $C_c(\mathbb{R}^d)$, $\forall \varepsilon > 0$, $\exists\, \varphi \in \mathcal{F}$ s.t. $\text{(I)} + \text{(II)} < \varepsilon$ $\forall k$.

Sending $k \to +\infty$ gives the result. $\square$

Sets of Finite Perimeter

In the last round of presentations, we will hear about the Gauss-Green Theorem, that, for all open sets E with C^1 boundary,

$$\int_E \nabla\cdot T(x)\,\underbrace{dx}_{\text{let } dx = d\mathcal{L}^d(x)} = \int_{\partial E} T\cdot \overset{\text{outer unit normal}}{\nu_E}\, d\mathcal{H}^{n-1}, \quad \forall T \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$$

The perimeter of E is given by

$$\mathcal{H}^{n-1}(\partial E) \overset{\uparrow}{=} |\mu_E|(\mathbb{R}^d).$$

If we define $\mu_E := \nu_E\, d\mathcal{H}^{n-1}\big|_{\partial E}$

Goal: generalize the notion of perimeter beyond sets with C^1 boundary.

Why more general E?

Sets with C^1 boundary can converge to sets without C^1 boundary.

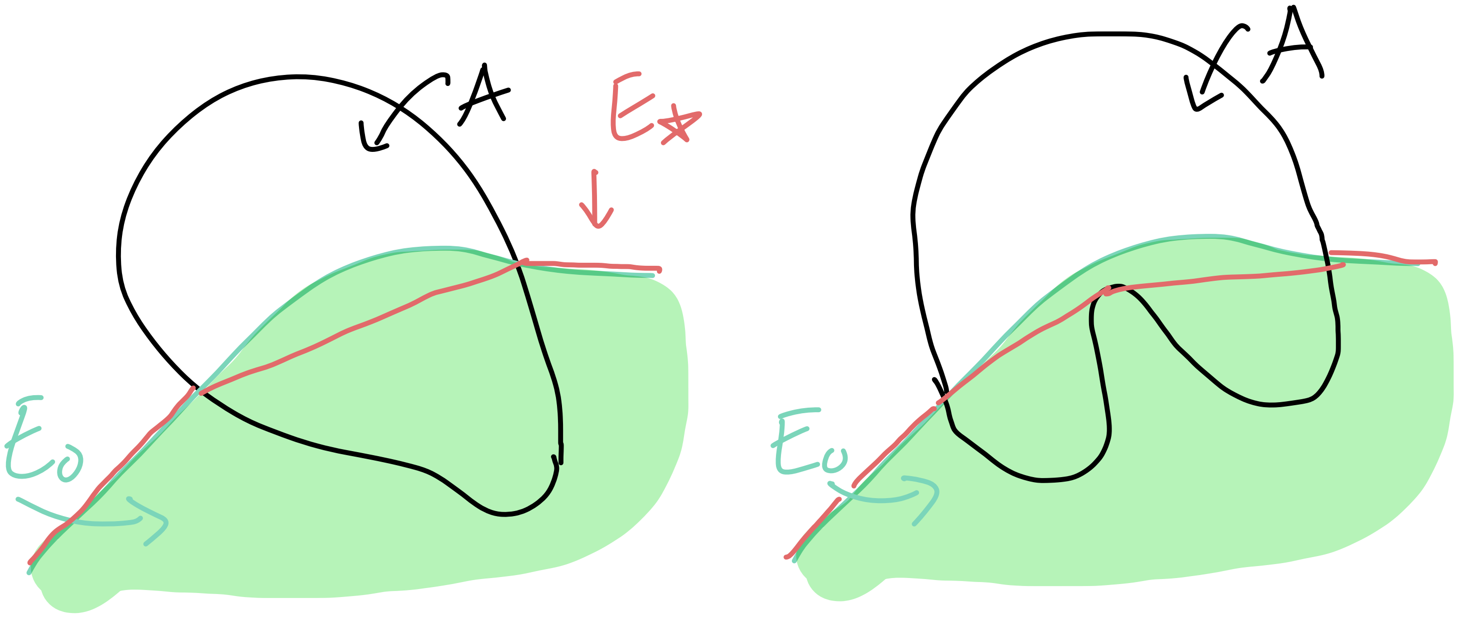
To study geometric optimization problems need to take limits of "nice" E to find an optimizer.

Examples of Geometric Variational Problems:

Plateau Problem

Given $A, E_0 \in \mathbb{R}^d$ where E_0 has finite perimeter

$$\inf_E \{ P(E) : E \setminus A = E_0 \setminus A \}$$



$E_0 \cap \partial A$ is "boundary condition" for E.

The Plateau problem is a prototype "minimal surface" problem.

Potential Energies

$$\inf_{E\subseteq A}\left\{P(E)+\int_E g(x)\,d\mathcal{L}(x)\right\}$$

Applications: soap films, membranes, equilibrium shape of liquid in container, materials science, general relativity...

Def: $E \in \mathcal{B}_{\mathbb{R}^d}$ has locally finite perimeter if $\exists$ locally finite vv measure μ_E so that

$$\int_E \nabla \cdot T(x)\,dx = \int_{\mathbb{R}^d} T \cdot d\mu_E, \quad \forall T \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$$

The perimeter of E is

$$P(E) := |\mu_E|(\mathbb{R}^d).$$

The relative perimeter of E in $\overset{\mathcal{B}_{\mathbb{R}^d}}{\overset{\rotatebox{90}{$\in$}}{F}}$,

$$P(E;F) = |\mu_E|(F).$$