Lecture 15

260R, Advanced Measure Theory

Reminder: Course Evaluations

<u>Goal</u>: generalize the notion of perimeter beyond sets with C^1 boundary.

<u>Def</u>: $E \in \mathcal{B}_{\mathbb{R}^d}$ has <u>locally finite perimeter</u> if $\exists$ locally finite vv measure μ_E so that

$$\underbrace{\int_E \nabla \cdot T(x) \overbrace{dx}^{d\mathcal{L}^d(x)}}_{\int_E \nabla\cdot T} = \int_{\mathbb{R}^d} T \cdot d\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$$

equiv to $\int_E \nabla\varphi = \int_{\mathbb{R}^d} \varphi \, d\mu_E \quad \forall \varphi \in C_c^1(\mathbb{R}^d)$

We call μ_E the <u>Gauss-Green measure</u> of E.

The perimeter of E is

$$P(E) := |\mu_E|(\mathbb{R}^d).$$

The relative perimeter of E in F is $\leftarrow \mathcal{B}_{\mathbb{R}^d}$

$$P(E;F) = |\mu_E|(F).$$

<u>Rmk</u>: If $E \subseteq \mathbb{R}^d$ has C^1 boundary,

$$\mu_E = \underbrace{\nu_E}_{\text{outer unit normal}} \, d\mathcal{H}^{d-1}\big|_{\partial E}$$

Thus,

$$P(E) = \mathcal{H}^{d-1}(\partial E)$$

$$P(E;F) = \mathcal{H}^{d-1}(\partial E \cap F)$$

<u>Rmk</u>: By defn, $\forall\, T \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \nabla\cdot T \, \mathbb{1}_E = \int T \, d\mu_E$$

$\Rightarrow$ often interpret $\mu_E = -\nabla \mathbb{1}_E$
(as distributional gradient)

Prop: $E \in \mathcal{B}_{\mathbb{R}^d}$ has locally finite perimeter iff, $\forall K \subseteq \mathbb{R}^d$ compact,

$$\sup\left\{\int_E \nabla\cdot T : T\in C_c^\infty(K;\mathbb{R}^d), |T|\leq 1\right\} < +\infty$$

Fact: $C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ is dense in $C_c(\mathbb{R}^d;\mathbb{R}^d)$ w.r.t. the inductive limit topology.

Pf: If E has locally finite perimeter

$$\sup\left\{\int_E \nabla\cdot T : T\in C_c^\infty(K;\mathbb{R}^d), |T|\leq 1\right\}$$

$$= \sup\left\{\int_{\mathbb{R}^d} T\cdot d\mu_E : T\in C_c^\infty(K;\mathbb{R}^d), |T|\leq 1\right\}$$

$$\leq |\mu_E|(K) < +\infty$$

Now, suppose that, $\forall K \subseteq \mathbb{R}^d$ compact

$$\sup\left\{ \int_E \nabla \cdot T : T \in C_c^\infty(K; \mathbb{R}^d), |T| \leq 1 \right\} < +\infty$$

Define $L : C_c^\infty(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}$ by

$$\langle L, T \rangle = \int_E \nabla \cdot T$$

By hypothesis, $\forall$ K cpt, $\exists C_K > 0$ s.t.

$$\langle L, T \rangle \leq \|T\|_\infty C_K$$

Thus if $\{T_n\}_{n \in \mathbb{N}}, \{\tilde{T}_m\}_{m \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ have same limit, $\langle L, T_n - T_m \rangle \to 0$.

In this way, L can be extended by density to a bounded linear functional on $C_c(\mathbb{R}^d; \mathbb{R}^d)$.

By Riesz-Representation, E has to finite perimeter. $\square$

Lower Semicontinuity

Def: Given $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}_{\mathbb{R}^d}$, $E \in \mathcal{B}_{\mathbb{R}^d}$, we say E_n locally converges to E if

$$\lim_{n \to \infty} \underbrace{\mathcal{L}^d(K \cap (E_n \triangle E))}_{|K \cap (E_n \triangle E)|} = 0 \quad \forall K \text{ cpt}$$

Rmk: Note that $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathbb{R}^d}$ and $E \in \mathcal{M}_{\mathcal{L}^d}(\mathbb{R}^d)$ it is possible that $E_n \xrightarrow{loc} E$ and $E \notin \mathcal{B}_{\mathbb{R}^d}$. However, there always exists $F \in \mathcal{B}_{\mathbb{R}^d}$ so that $|E \Delta F| = 0$, so $E_n \xrightarrow{loc} F$.

Prop: Given $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathbb{R}^d}$ with locally finite perimeter s.t. $\exists$ $E \in \mathcal{B}_{\mathbb{R}^d}$ $E_n \xrightarrow{loc} E$, $\limsup_{n\to\infty} P(E_n; K) < +\infty$ $\forall K$ cpt,

then

(i) E has locally finite perimeter

(ii) $\mu_{E_n} \to \mu_E$ vaguely

(iii) $\forall A$ open, $\liminf_{n\to\infty} P(E_n; A) \geq P(E; A)$

Rmk: If A open and F has locally finite perimeter,

Riesz Representation

$$P(F;A) = |\mu_F|(A)$$

$$= \sup\{\langle L_{\mu_F}, T\rangle : T \in C_c(A;\mathbb{R}^d),\ |T| \leq 1\}$$

$$= \sup\left\{\int_F \nabla\cdot T : T \in C_c^\infty(A;\mathbb{R}^d),\ |T| \leq 1\right\}$$

Pf: Note that $\forall\, T \in C_c^\infty(A;\mathbb{R}^d)$, $|T| \leq 1$, $E_n \xrightarrow{loc} E$, $\left|\int_{E_n} \varphi - \int_E \varphi\right| \leq \|\varphi\|_\infty |E_n \Delta E|$

$$\liminf_{n\to\infty} P(E_n, A) \geq \liminf_{n\to\infty} \int_{E_n} \nabla\cdot T = \int_E \nabla\cdot T$$

For any K cpt, $\exists\, R>0$ s.t. $K \subseteq B_R \subseteq \overline{B_R}$

Thus, $\forall\, T \in C_c^\infty(K;\mathbb{R}^d)$, $|T| \leq 1$,

$$\int \nabla\cdot T \, \mathbb{1}_E \leq \liminf_{n\to\infty} P(E_n; \overline{B_R}) < +\infty$$

Hence, by prev prop, E has locally finite perimeter, so (i).

Now, using ☆ and previous remark gives (iii)

Finally, using ☆, $A = \mathbb{R}^d$
$\forall\, T \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

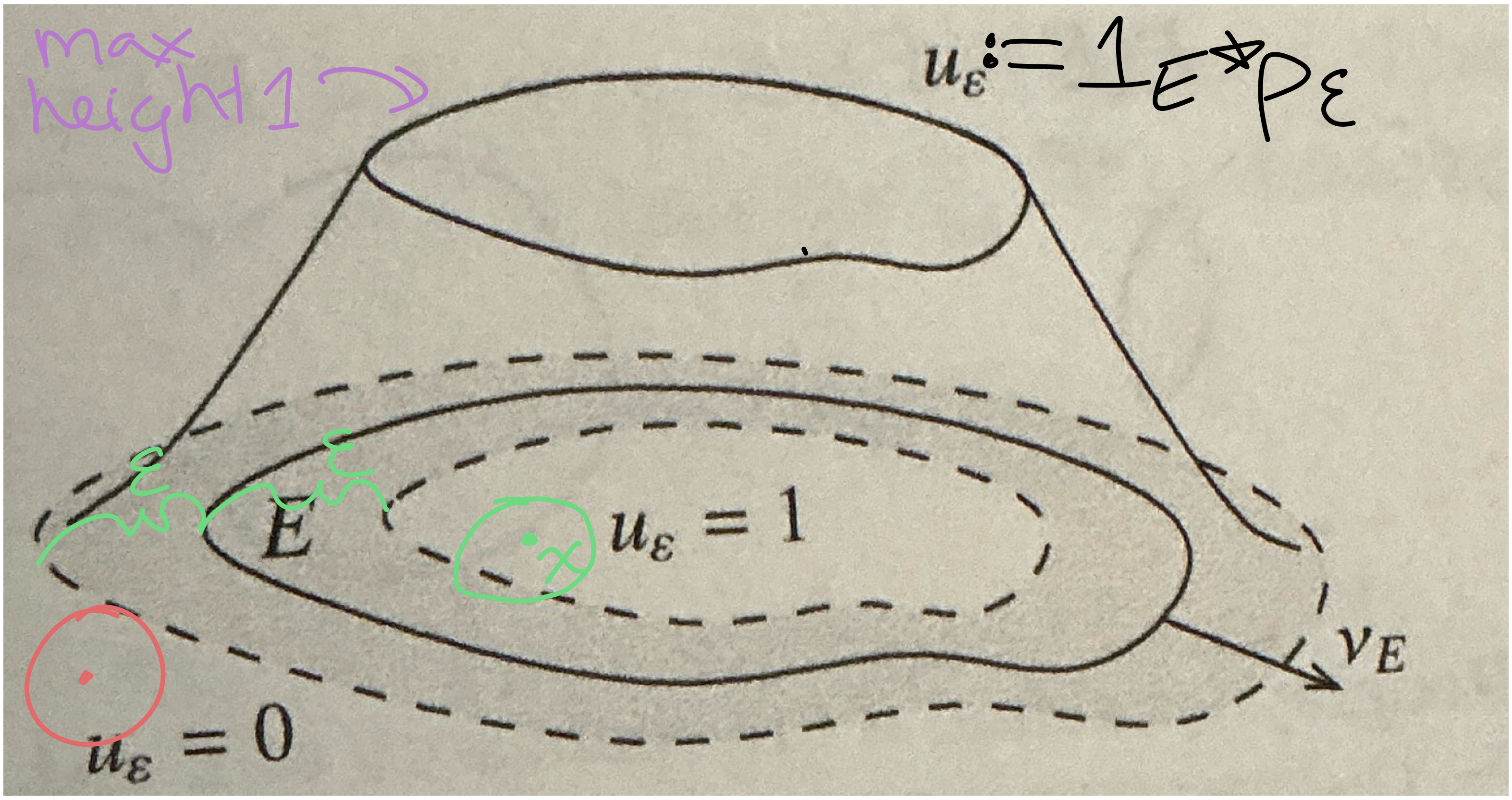
$$\lim_{n\to\infty} \int T\, d\mu_{E_n} = \lim_{n\to\infty} \int_{E_n} \nabla\cdot T = \int_E \nabla\cdot T = \int T\, d\mu_E$$

By density of $C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ in $C_c(\mathbb{R}^d;\mathbb{R}^d)$, $\mu_{E_n} \to \mu$ vaguely. $\square$

Approximation of μ_E via convolution with mollifiers...

Warmup:
Fix $E \in \mathcal{B}_{\mathbb{R}^d}$, $\rho \in C_c^\infty(B_1;[0,+\infty))$, even, $\int \rho = 1$; $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)$.

$$(1_E * \rho_\varepsilon)(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(x-y) 1_E(y)\,dy$$
$$= \int_E \rho_\varepsilon(x-y)\,dy$$



From the picture, we guess...

$$\nabla(1_E * \rho_\varepsilon)(x) \approx \frac{1}{\varepsilon}\nu_E(\pi_{\partial E}(x))$$

We expect...

$$\int |\nabla(1_E * \rho_\varepsilon)| \approx \frac{|\{y \in \mathbb{R}^d : \mathrm{dist}(y, \partial E) < \varepsilon\}|}{\varepsilon}$$

$$\approx \frac{\varepsilon \mathcal{H}^{d-1}(\partial E)}{\varepsilon} = P(E)$$

Prop: If $E \in \mathcal{B}_{\mathbb{R}^d}$ has locally finite perimeter, then

- $\nabla(1_E * \rho_\varepsilon) = \mu_E * \rho_\varepsilon \quad \forall\, \varepsilon > 0$
- $\nabla(1_E * \rho_\varepsilon)\mathcal{L}^d \to \mu_E$ vaguely
- $|\nabla 1_E * \rho_\varepsilon|\mathcal{L}^d \to |\mu_E|$ vaguely

Pf:

By equiv formulation of Gauss Green measure,

$$\mu_E * \rho_\varepsilon(x) = \int_{\mathbb{R}^d} \overbrace{\rho_\varepsilon(x-y)}^{\varphi(y)}\, d\mu_E(y)$$

$$= -\int_E \nabla \rho_\varepsilon(x-y)\, dy$$

$$= -\nabla(1_E * \rho_\varepsilon)(x)$$

Remaining bullets follow from Max presentation. $\square$

<u>Lemma</u>: If $E, F \in \mathcal{B}_{\mathbb{R}^d}$ have locally finite perimeter, then so do $E \cup F$ and $E \cap F$, and for all $A \subseteq \mathbb{R}^d$ open,

$$P(E \cup F; A) + P(E \cap F; A) \le P(E; A) + P(F; A)$$

<u>Pf</u>: Define

$$u_\varepsilon := 1_E * \rho_\varepsilon$$
$$v_\varepsilon := 1_F * \rho_\varepsilon$$

Then, in $L_{loc}(\mathbb{R}^d)$,

$$u_\varepsilon v_\varepsilon \to 1_{E \cap F}$$

$$\overbrace{u_\varepsilon + v_\varepsilon - u_\varepsilon v_\varepsilon}^{w_\varepsilon} \to 1_{E \cup F}$$

Furthermore, $\forall A \subseteq \mathbb{R}^d$ open, bounded

$$\int_A |\nabla(u_\varepsilon v_\varepsilon)| \leq \int_A v_\varepsilon |\nabla u_\varepsilon| + u_\varepsilon |\nabla v_\varepsilon|$$

$$\int_A |\nabla w_\varepsilon| \leq \int_A (1-v_\varepsilon)|\nabla u_\varepsilon| + (1-u_\varepsilon)|\nabla v_\varepsilon|$$

$\downarrow$ adding

$$\int_A |\nabla(u_\varepsilon v_\varepsilon)| + \int_A |\nabla w_\varepsilon| \leq \int_A |\nabla u_\varepsilon| + |\nabla v_\varepsilon|$$

Since A bdd, $\overline{A}$ cpt

$\nu_n \to \nu$ vaguely

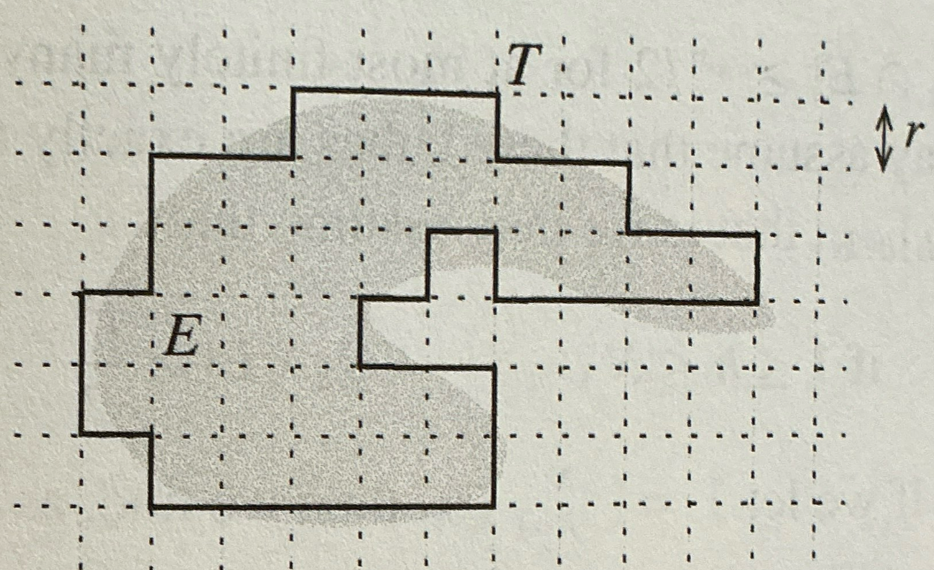
$\forall K$ cpt $\limsup_{n\to\infty} \nu_n(K) \leq \nu(K)$

$$\limsup_{\varepsilon\to 0} \int_A |\nabla u_\varepsilon| + |\nabla v_\varepsilon| \leq \limsup_{\varepsilon\to 0} \int_{\overline{A}} |\nabla u_\varepsilon| + |\nabla v_\varepsilon|$$

$$\leq |\mu_E|(\overline{A}) + |\mu_F|(\overline{A})$$

$$= P(E;\overline{A}) + P(F;\overline{A})$$

finish next time :)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