Lecture 16

260R, Advanced Measure Theory

Reminder: Course Evaluations

Def: $E \in \mathcal{B}_{\mathbb{R}^d}$ has locally finite perimeter if $\exists$ locally finite $\mathbb{R}^d$ measure μ_E so that

$$\int_E \nabla \cdot T(x)\,dx = \int_{\mathbb{R}^d} T \cdot d\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$$

We call μ_E the Gauss-Green measure of E.

The perimeter of E is

$$P(E) := |\mu_E|(\mathbb{R}^d).$$

The relative perimeter of E in F $\in \mathcal{B}_{\mathbb{R}^d}$ is

$$P(E;F) = |\mu_E|(F).$$

Prop: $E \in \mathcal{B}_{\mathbb{R}^d}$ has locally finite perimeter iff, $\forall K \subseteq \mathbb{R}^d$ compact,

$$\sup\left\{ \int_E \nabla\cdot T : T \in C_c^\infty(K;\mathbb{R}^d), |T| \leq 1 \right\} < +\infty$$

Fact: $C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ is dense in $C_c(\mathbb{R}^d;\mathbb{R}^d)$ w.r.t. the inductive limit topology.

Rmk: If A open and F has locally finite perimeter,

$P(F;A)$

$$= \sup\left\{ \int_F \nabla\cdot T : T \in C_c^\infty(A;\mathbb{R}^d),\ |T|\le 1 \right\}$$

Lower Semicontinuity

Def: Given $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathbb{R}^d}$, $E \in \mathcal{B}_{\mathbb{R}^d}$, we say E_n locally converges to E if

$$\lim_{n\to\infty} \underbrace{\mathcal{L}^d(K\cap(E_n \Delta E))}_{|K\cap(E_n\Delta E)|} = 0 \quad \forall K \text{ cpt}$$

Prop: Given $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathbb{R}^d}$ with locally finite perimeter s.t. $\exists\ E \in \mathcal{B}_{\mathbb{R}^d}$

$E_n \xrightarrow{\text{loc}} E$, $\limsup_{n\to\infty} P(E_n;K) < \infty$ $\forall K$ cpt,

then

(i) E has locally finite perimeter

(ii) $\mu_{E_n} \to \mu_E$ vaguely

(iii) $\forall A$ open, $\liminf_{n\to\infty} P(E_n;A) \geq P(E;A)$

Approximation of μ_E via convolution with mollifiers...

Warmup:

Fix $E \in \mathcal{B}_{\mathbb{R}^d}$, $\rho \in C_c^\infty(B_1;[0,+\infty))$, even, $\int \rho = 1$, $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)$.

Prop: If $E \in \mathcal{B}_{\mathbb{R}^d}$ has locally finite perimeter, then

- $\nabla(1_E * \rho_\varepsilon) = \mu_E * \rho_\varepsilon \quad \forall\, \varepsilon > 0$
- $\nabla(1_E * \rho_\varepsilon)\mathcal{L}^d \to \mu_E$ vaguely
- $|\nabla 1_E * \rho_\varepsilon|\mathcal{L}^d \to |\mu_E|$ vaguely

Lemma: If $E, F \in \mathcal{B}_{\mathbb{R}^d}$ have locally finite perimeter, then so do $E \cup F$ and $E \cap F$, and for all $A \subseteq \mathbb{R}^d$ open,

$$P(E \cup F; A) + P(E \cap F; A) \leq P(E; A) + P(F; A)$$

Pf: Define $u_\varepsilon := 1_E * \rho_\varepsilon$

$v_\varepsilon := 1_F * \rho_\varepsilon$

Then, in $L_{loc}(\mathbb{R}^d)$,

$$u_\varepsilon v_\varepsilon \to 1_{E\cap F}$$

$$\overbrace{u_\varepsilon + v_\varepsilon - u_\varepsilon v_\varepsilon}^{w_\varepsilon} \to 1_{E\cup F}$$

Furthermore, $\forall A \subseteq \mathbb{R}^d$ open, bounded

$$\int_A |\nabla(u_\varepsilon v_\varepsilon)| \leq \int_A v_\varepsilon |\nabla u_\varepsilon| + u_\varepsilon |\nabla v_\varepsilon|$$

$$\int_A |\nabla w_\varepsilon| \leq \int_A (1-v_\varepsilon)|\nabla u_\varepsilon| + (1-u_\varepsilon)|\nabla v_\varepsilon|$$

$\downarrow$ adding

$$\int_A |\nabla(u_\varepsilon v_\varepsilon)| + \int_A |\nabla w_\varepsilon| \leq \int_A |\nabla u_\varepsilon| + |\nabla v_\varepsilon|$$

Since A bdd, $\bar{A}$ cpt

$\nu_n \to \nu$ vaguely
$\forall K$ cpt $\limsup_{n\to\infty} \nu_n(K) \le \nu(K)$

$$\limsup_{\varepsilon\to 0}\int_A |\nabla u_\varepsilon|+|\nabla v_\varepsilon| \le \limsup_{\varepsilon\to 0}\int_{\bar{A}} |\nabla u_\varepsilon|+|\nabla v_\varepsilon|$$
$$\le |\mu_E|(\bar{A}) + |\mu_F|(\bar{A})$$
$$= P(E;\bar{A}) + P(F;\bar{A})$$

For the LHS, note that, $\forall\, T \in C_c^\infty(A;\mathbb{R}^d)$, $|T| \le 1$,

$$\liminf_{\varepsilon\to 0}\int_A |\nabla(u_\varepsilon v_\varepsilon)|$$
$$\ge \liminf_{\varepsilon\to 0} -\int T\,\nabla(u_\varepsilon v_\varepsilon)$$
$$= \liminf_{\varepsilon\to 0}\int \nabla\cdot T\,(u_\varepsilon v_\varepsilon)$$
$$= \int_{E\cap F} \nabla\cdot T$$

By ☼,

$$\liminf_{\varepsilon \to 0} \int_A |\nabla(u_\varepsilon v_\varepsilon)| \geq P(E \cap F; A)$$

Similarly

$$\liminf_{\varepsilon \to 0} \int_A |\nabla w_\varepsilon| \geq P(E \cup F; A)$$

Thus, for A open and bdd,

ⓜ $$P(E;\bar{A}) + P(F;\bar{A}) \geq P(E \cap F; A) + P(E \cup F; A)$$

Fix arbitrary A open.
Define $A_k := \{x \in A \cap B_k : \mathrm{dist}(x, \partial A) > \frac{1}{k}\}$
These A_k are open + bdd.

Furthermore...

$\overline{A}_k \subseteq A$, $A_k \subseteq A_{k+1}$, $\bigcup_k A_k = A$

By

$$P(E;A)+P(F;A)$$

$$\geq P(E;\overline{A}_k)+P(F;\overline{A}_k)$$

$$\geq P(E\cap F;A_k)+P(E\cup F;A_k)$$

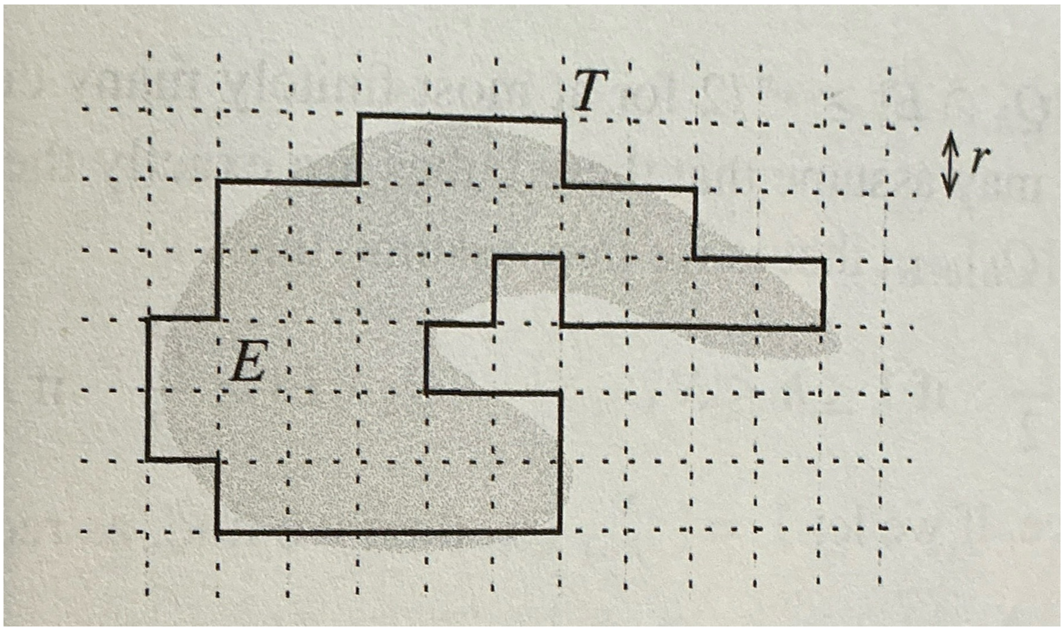
The result then follows by continuity from below. $\square$

We have lower semicontinuity, we can approximate via mollification, ... compactness!

Lemma: Let $\{Q_n\}_{n\in\mathbb{N}}$ be a disjoint family of open cubes of side length $r>0$ s.t $\bigcup_{n\in\mathbb{N}} Q_n = \mathbb{R}^d$.

Then, for any $E\in\mathcal{B}_{\mathbb{R}^d}$ bounded and with finite perimeter, $\exists$ finite, disjoint subcollection, $\{Q_n\}_{n\in I}$ s.t.

$$|E \triangle \bigcup_{n\in I} Q_n| \leq \sqrt{d}\, r\, P(E).$$



The proof of the lemma relies on the following fact:

Fact: For $Q := x + (0,r)^d$, $u \in C^1(\mathbb{R}^d)$,

$$\int_Q |u - (u)_Q| \le \sqrt{d}\, r \int_Q |\nabla u| .$$

$$(u)_Q := \frac{1}{|Q|} \int_Q u(y)\, dy$$

Pf of Lemma:

Fix $\varepsilon > 0$, by Fact for $u = 1_E * \rho_\varepsilon$,

$$\int_{\mathbb{R}^d \cap \overline{B}_R} |\nabla 1_E * \rho_\varepsilon| = \sum_n \int_{Q_n} |\nabla 1_E * \rho_\varepsilon|$$

$$\geq \frac{1}{\sqrt{d}\, r} \sum_n \int_{Q_n} |1_E * \rho_\varepsilon - (1_E * \rho_\varepsilon)_{Q_n}|$$

Since E bdd, $\exists\, R > 0$ s.t. $\operatorname{supp} \nabla 1_E * \rho_\varepsilon \subseteq \overline{B}_R$.

Taking $\limsup_{\varepsilon \to 0}$ LHS $\leq |\mu_E|(\overline{B}_R)$

$\leq |\mu_E|(\mathbb{R}^d)$

$= P(E)$.

Taking $\liminf_{\varepsilon \to 0}$ RHS...

$$\geq \frac{1}{\sqrt{d}\, r} \sum_{n} \underbrace{\int_{Q_n} \left| 1_E - (1_E)_{Q_n} \right|}$$

$$= \int_{Q_n} \left| 1_E - \frac{|E \cap Q_n|}{r^d} \right|$$

$$= \left| 1 - \frac{|E \cap Q_n|}{r^d} \right| |Q_n \cap E| + \frac{|E \cap Q_n|}{r^d} |Q_n \setminus E|$$

$$= 2 \frac{|E \cap Q_n|}{r^d} |Q_n \setminus E|$$

if $|Q_n \cap E| < \frac{r^d}{2}$ then $|Q_n \setminus E| \geq \frac{r^d}{2}$

Since E is bdd, $|Q_n \cap E| \geq \frac{r^d}{2}$ for at most finitely many n. Let I be the indices of those Q_n. Let $T = \bigcup_{n \in I} Q_n$, continuing the above ineq...

$$\geq \frac{1}{\sqrt{d}\, r} \left(\sum_{n \in I} |Q_n \setminus E| + \sum_{n \notin I} |E \cap Q_n| \right)$$

$$= \frac{1}{\sqrt{d}\, r} \left(|T \setminus E| + |E \setminus T| \right) \qquad \square$$

Prop: Consider the set $\mathcal{X} := \{E \in \mathcal{B}_{\mathbb{R}^d} : |E| < +\infty\}$ endowed with the distance fn $d(E,F) := |E \Delta F|$.

Then $(\mathcal{X}, d)$ is a complete metric space and for any $R, p > 0$, $\mathcal{Y}_{R,p} := \{E \in \mathcal{B}_{\mathbb{R}^d} : E \subseteq B_R, P(E) \leq p\}$ is compact.

Rmk: Convergence in d implies local convergence.

Rmk: We identify $E \sim F$ if $|E \Delta F| = 0$.

Pf: We will show $\mathcal{Y}_{R,p}$ is compact.

Suppose $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Y}_{R,p}$ satisfies $E_n \xrightarrow{d} E$. By prev prop, $P(E) \leq p$. Also $E \subseteq B_R$ (up to sets of Lebesgue measure zero). Thus $E \in \mathcal{Y}_{R,p}$, so $\mathcal{Y}_{R,p}$ is closed.

It remains to show $\mathcal{Y}_{R,p}$ is totally bounded.

Fix $\sigma > 0$. We will show $\exists \{T_m\}_{m=1}^{M} \subseteq \mathcal{X}$ s.t.

$$\min_{m=1,\dots,M} d(E, T_m) < \sigma \qquad \forall E \in \mathcal{X}$$

Choose $r > 0$ s.t. $\sqrt{d}\,rp < \sigma$

By our previous lemma, for any $E \in \mathcal{Y}_{R,p}$, $\exists$ finite subcollection I s.t.

$$d\Big(E, \bigcup_{n \in I} Q_n\Big) < \sigma$$

Since $E \subseteq B_R$, WLOG, we may take only $Q_n \subseteq B_R$.

There are only finitely many ways to take unions of $Q_n \subseteq B_R$. $\square$

Thm: Suppose $\{E_n\}_{n\in\mathbb{N}}$ are sets of finite perimeter s.t.

- $\sup_{n\in\mathbb{N}} P(E_n) < +\infty$,
- $\exists R$ s.t. $E_n \subseteq B_R$ $\forall n \in \mathbb{N}$.

Then there exists a set $E \in \mathcal{B}_{\mathbb{R}^d}$ with finite perimeter s.t. $E \subseteq B_R$ and $\exists$ subsequence n_k s.t.

(i) $E_{n_k} \xrightarrow{d} E$

(ii) $\mu_{E_{n_k}} \rightarrow \mu_E$ vaguely

Pf: Let $p := \sup_{n\in\mathbb{N}} P(E_n)$. Then $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Y}_{R,p}$. So compactness ensures $\exists$ $E \in \mathcal{Y}_{R,p}$ and $E_{n_k} \rightarrow E$.

Vague convergence $\mu_{E_{n_k}} \rightarrow \mu_E$ follows from Prop on lsc of perimeter.

Plateau Problem

Thm: Given $A \subseteq \mathbb{R}^d$ bounded and $E_0 \in \mathcal{B}_{\mathbb{R}^d}$ with finite perimeter, there exists a minimizer of

$$\min_{E \in \mathcal{B}_{\mathbb{R}^d}} \{ P(E) : E \setminus A = E_0 \setminus A \}$$

Pf: Define

$$\gamma(A, E_0) := \inf_{E \in \mathcal{B}_{\mathbb{R}^d}} \{P(E) : E \setminus A = E_0 \setminus A\}$$

Since E_0 satisfies constraints and $P(E_0) < +\infty$, we have $\gamma(A, E_0) < +\infty$.

Thus, we may take a minimizing sequence E_n. We may assume $P(E_n) \le P(E_0)$.
Define $M_n := E_n \triangle E_0$
$= (E_n \setminus E_0) \cup (E_0 \setminus E_n)$.

By Prop,

$$\begin{aligned} P(M_n) &\le P(E_n \cap E_0^c) + P(E_0 \cap E_n^c) \\ &\le P(E_n) + P(E_0^c) + P(E_0) + P(E_n^c) \\ &\le 2P(E_n) + 2P(E_0) \le 4P(E_0). \end{aligned}$$

$P(E_0^c) = P(E_0) \downarrow$

Since E_0 satisfies constraints, $\gamma(A, E_0) \le P(E_0) < +\infty$.

Take a minimizing sequence E_n, and let

$$M_n := E_n \triangle E_0 = (E_n \setminus E_0) \cup (E_0 \setminus E_n$$

Then, by Prop,

$$P(M_n) \le P(E_n \setminus E_0) + P(E_0 \setminus E_n)$$

$P(F) = P(F^c)$

$$\le P(E_n) + P(E_0^c) + P(E_0) + P(E_n^c)$$

$$= 2P(E_n) + 2P(E_0)$$

$$\le 4P(E_0)$$

Since $M_n \subseteq A$, A bdd, $\exists$ M with finite perimeter s.t., up to a subsequence, $M_n \to M$.

Furthermore, $M_n \subseteq A$ and A is bounded.

Thus, $\exists\, M \in \mathcal{B}_{\mathbb{R}^d}$ with finite perimeter s.t $M_{n_k} \to M$.

Since $E_n = (E_0 \cup M_n) \setminus (E_0 \cap M_n)$ we have

$$E_{n_k} \to \underbrace{(E_0 \cup M) \setminus (E_0 \cap M)}_{=: E}.$$

Finally, by def, $E \setminus A = E_0 \setminus A$ and lsc of perimeter ensures

$$\gamma(A, E_0) = \liminf_{k \to \infty} P(E_{n_k}) \geq P(E)$$

Thus E is a minimizer. $\square$