Lecture 1

260R, Advanced Measure Theory

Course Outline

I. Review and Hausdorff Measure

II. Signed Measures and Differentiation
- A. Jordan-Hahn Decomposition
- B. Radon-Nikodym Theorem
- C. Lebesgue Differentiation Theorem
- D. Fundamental Theorem of Calculus

III. Radon Measures
- A. Riesz Representation Theorem
- B. Weak-* convergence
- C. Prokhorov's Theorem

IV. Sets of finite perimeter

A. Geometric Variational Problems

B. Isoperimetric Inequality

Review: Measure Theory

Folland Ch. 1, Maggi Ch. 1-2

Outer Measures

Fix a nonempty set X.

Def: An <u>outer measure</u> μ^* on X is a function $\mu^*: 2^X \to [0,+\infty]$ s.t.

(i) $\mu^*(\emptyset) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Aside: Hausdorff measure on $\mathbb{R}^d$

Def: Fix $d \in \mathbb{N}$, $k > 0$, $\delta > 0$. ($k \in \mathbb{N}$ most of the time)
The (outer) <u>k-dimensional Hausdorff measure of step</u> δ on $\mathbb{R}^d$ is

$$\mathcal{H}^k_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} \omega_k \left(\frac{\operatorname{diam}(F_i)}{2}\right)^k : \{F_i\}_{i=1}^{\infty} \text{ s.t. } E \subseteq \bigcup_{i=1}^{\infty} F_i,\ \operatorname{diam}(F_i) < \delta \right\}$$

for any $E \subseteq \mathbb{R}^d$

<u>Glossary</u>:

$\omega_k = \pi^{k/2} / \Gamma(1 + \frac{k}{2}) \overset{k \in \mathbb{N}}{=} \lambda^k(B_1(0))$

On a metric space (X, d), $F \subseteq X$,

$\operatorname{diam}(F) = \sup\{d(x,y) : x, y \in F\}$

$\star$ = k-dim'l area of F_i

Rmk: $\mathcal{H}^k_\delta$ is an outer measure (201a, HW3, Q4)

Def: Fix $d \in \mathbb{N}$, $k > 0$. The (outer) <u>k-dimensional Hausdorff measure</u> is

$$\mathcal{H}^k(E) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E) = \sup_{\delta > 0} \mathcal{H}^k_\delta(E)$$

Lemma:

(i) $\mathcal{H}^k$ is an outermeasure

(ii) $\mathcal{H}^k$ is translation invariant

(iii) $\mathcal{H}^k(\underbrace{\alpha E}_{\{\alpha x \,:\, x \in E\}}) = \alpha^k \mathcal{H}^k(E)$, $\forall \alpha > 0$.

Pf: Follow from defn

Key Properties of Hausdorff Measure

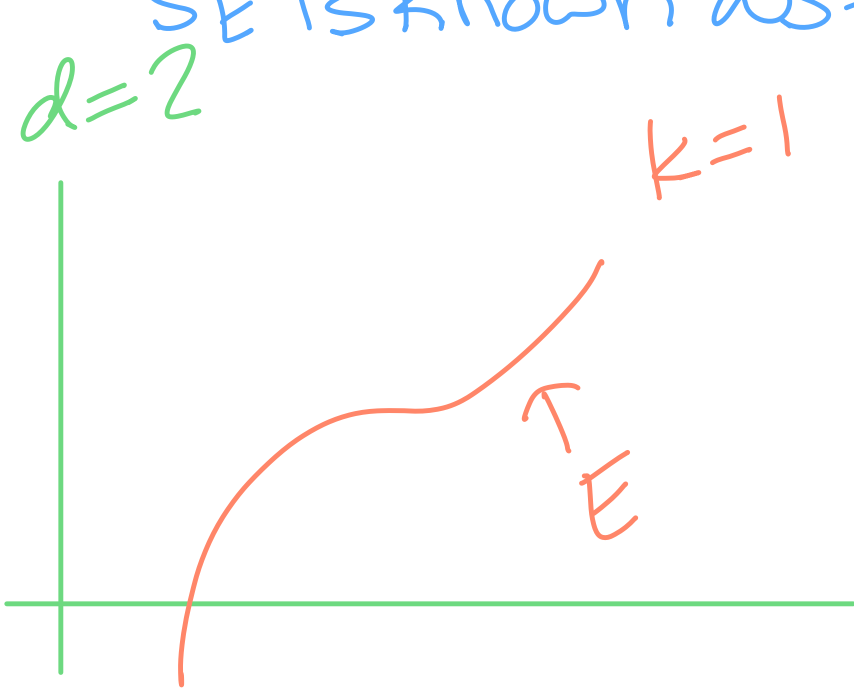
(i) For $k \in \mathbb{N}$, $1 \leq k \leq d$, and E a k-dimensional parametrized surface,
$\mathcal{H}^k(E) = \text{area}(E)$

(ii) If $k > d$, $\mathcal{H}^k(E) = 0$, $\forall E \subseteq \mathbb{R}^d$

(iii) More genereally, for all $E \subseteq \mathbb{R}^d$, $\exists\, s_E \geq 0$ s.t.

$$\mathcal{H}^k(E) = \begin{cases} +\infty & \text{for } k < s_E \\ 0 & k > s_E \end{cases}$$

s_E is known as the <u>Hausdorff dimension</u> of E.



(iv) $\mathcal{H}^d = \lambda^d$

Measures

σ-algebra,
closed under complements
and countable unions

Measurable space $(X, \mathcal{M})$

"positive"

<u>Def</u>: A <u>measure</u> μ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, +\infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) μ is countably additive.

<u>Thm</u> (Carathéodory): Given an outer measure μ^*, define

$$\mathcal{M}_{\mu^*} = \{A \subseteq X : \overbrace{\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)}^{(\star)} \ \forall E \subseteq X\}$$

Then $\mathcal{M}_{\mu^*}$ is a σ-algebra and $\mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure.

To show Hausdorff measure is a Borel measure, we will use the following criterion:

<u>Thm</u>: Suppose (X,d) is a metric space and μ^* is an outer measure on X.

Then $\mu^*|_{\mathcal{B}(X)}$ is a measure iff

$\boxed{\forall E, F \subseteq X,\ \mathrm{dist}(E,F) > 0, \\ \mu^*(E \cup F) = \mu^*(E) + \mu^*(F).}$

"μ^* is a metric outer measure"

Glossary

$\mathrm{dist}(E, F) = \inf\{d(x,y) : x \in E, y \in F\}$

Pf: First, suppose $\mu^*|_{\mathcal{B}(X)}$ is a measure. Fix $E, F \subseteq X$ with $\mathrm{dist}(E,F) > 0$. Then $\mathrm{dist}(\overline{E}, F) > 0$. Then

{$A = \overline{E}$

$$\mu^*(E \cup F) = \mu^*((E \cup F) \cap \overline{E}) + \mu^*((E \cup F) \cap (\overline{E})^c)$$

$= \mu^*(E) + \mu^*(F)$.

Thus, μ^* is a metric outer measure.

Now, suppose μ^* is a metric outer measure.
It suffices to show that, for $A \subseteq X$ closed, $A \in \mathcal{M}_{\mu^*}$.
This guarantees $\mathcal{B}(X) \subseteq \mathcal{M}_{\mu^*}$, so $\mu^*|_{\mathcal{B}(X)}$ is a measure.

Fix $E \subseteq X$ with $\mu^*(E) < +\infty$.
We will show (A) holds with "$\geq$".

For $n \in \mathbb{N}$, define
$A_n = \{x \in E : \frac{1}{n+1} \leq \text{dist}(x, A) < \frac{1}{n}\}$

$A_0 = \{x \in E : 1 \le \operatorname{dist}(x, A)\}$.

By defn $\underbrace{A_0 \cup \left(\bigcup_{h \in \mathbb{N}} A_h\right)}_{\text{disjoint union}} \subseteq E \cap A^c$

Since A^c is open, the reverse containment also holds.

$$\mu^*(E \cap A) + \mu^*(E \cap A^c)$$

subadditivity of μ^*

$$\le \mu^*(E \cap A) + \mu^*\left(\bigcup_{h=0}^{N} A_h\right) + \sum_{h=N+1}^{\infty} \mu^*(A_h)$$

metric outer measure

$$= \mu^*\left((E \cap A) \cup \left(\bigcup_{h=0}^{N} A_h\right)\right) + \sum_{h=N+1}^{\infty} \mu^*(A_h)$$

$$\le \mu^*(E) + \sum_{h=N+1}^{\infty} \mu^*(A_h)$$

Again, using metric outer measure property,

$$\sum_{n=1}^{N} \mu^*(A_{2n}) = \mu^*\left(\bigcup_{n=1}^{N} A_{2n}\right) \le \mu^*(E) < +\infty.$$

$$\sum_{n=1}^{N} \mu^*(A_{2n-1}) = \mu^*\left(\bigcup_{n=1}^{N} A_{2n-1}\right) \le \mu^*(E)$$

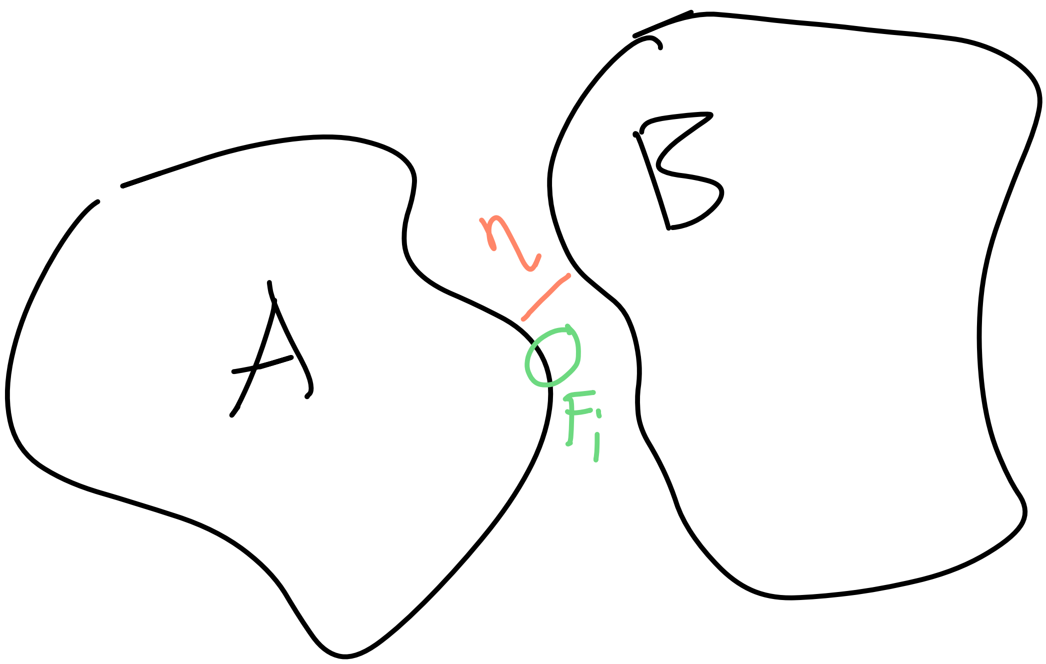
$+\infty$

Thus, both series converge, so the tail $\to 0$. $\square$

Prop: Fix $d \in \mathbb{N}$, $k \in (0,d)$. $\mathcal{H}^k$ is a Borel measure on $\mathbb{R}^d$.

Pf: Fix $A, B \subseteq \mathbb{R}^d$, $\operatorname{dist}(A,B) = \eta > 0$.

Then, for all $\delta < \eta/2$, any F_i with $\text{diam}(F_i) < \delta$ can intersect at most one of A or B.



Thus, every cover of $A \cup B$ gives disjoint covers of A and B and vice versa.

$$\mathcal{H}^k_\delta(A \cup B) = \mathcal{H}^k_\delta(A) + \mathcal{H}^k_\delta(B)$$

Sending $\delta \to 0$ gives

$\mathcal{H}^k(A \cup B) = \mathcal{H}^k(A) + \mathcal{H}^k(B)$.

Since $\mathcal{H}^k$ is a metric outer measure, so by prev thm, $\mathcal{H}^k\big|_{\mathcal{B}_{\mathbb{R}^d}}$ is a measure. $\square$

Exercise: $\forall\, d \in \mathbb{N}$, $k \in (0,d)$, $\delta > 0$, $\mathcal{H}^k_\delta$ is <u>not</u> a Borel measure on $\mathbb{R}^d$.