Lecture 2

260R, Advanced Measure Theory

Reminder: Presentation Topics (Apr 16)
Makeup Lecture: Friday, April 10th
2-3:45 pm

Outer Measures

Fix a nonempty set X.

Def: An <u>outer measure</u> μ^* on X is a function $\mu^*: 2^X \to [0,+\infty]$ s.t.

(i) $\mu^*(\emptyset) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \mu^*(A_i)$.

Def: Fix $d \in \mathbb{N}$, $k > 0$, $\delta > 0$.
The (outer) <u>k-dimensional Hausdorff measure of step δ</u> on $\mathbb{R}^d$ is

$$\mathcal{H}^k_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} \omega_k \left(\frac{\operatorname{diam}(F_i)}{2} \right)^k : \{F_i\}_{i=1}^{\infty} \text{ s.t. } E \subseteq \bigcup_{i=1}^{\infty} F_i, \ \operatorname{diam}(F_i) < \delta \right\}$$

for any $E \subseteq \mathbb{R}^d$.

Rmk: $\mathcal{H}^k_\delta$ is an outer measure (201a, HW3, Q4)

Def: Fix $d \in \mathbb{N}$, $k > 0$. The (outer) **k-dimensional Hausdorff measure** is

$$\mathcal{H}^k(E) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E) = \sup_{\delta > 0} \mathcal{H}^k_\delta(E)$$

Lemma:

(i) $\mathcal{H}^k$ is an outer measure

(ii) $\mathcal{H}^k$ is translation invariant

(iii) $\mathcal{H}^k(\alpha E) = \alpha^k \mathcal{H}^k(E)$, $\forall \alpha > 0$.

Key Properties of Hausdorff Measure

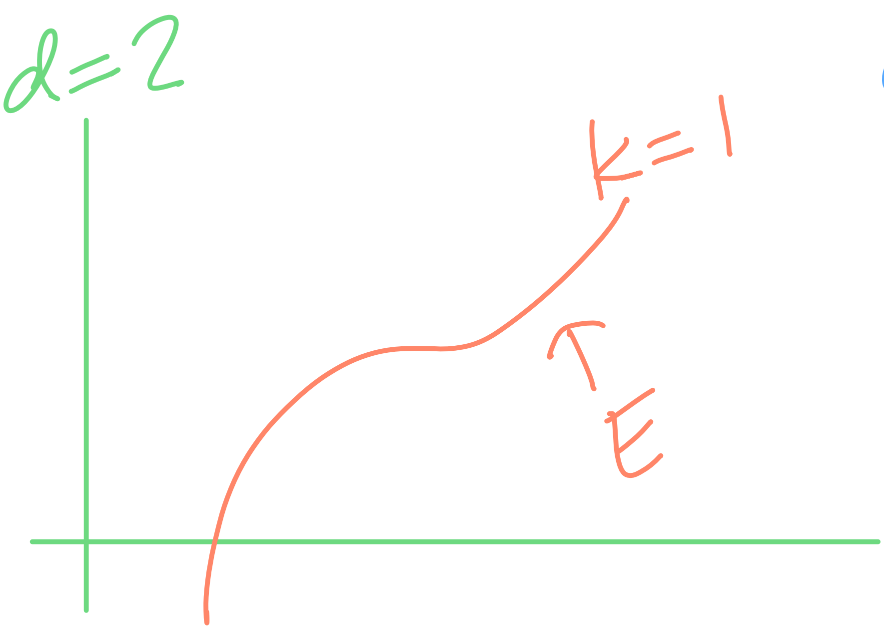
(i) For $k \in \mathbb{N}$, $1 \leq k \leq d$, and E a k-dimensional parametrized surface,
$\mathcal{H}^k(E) = \text{area}(E)$

(ii) If $k > d$, $\mathcal{H}^k(E) = 0$, $\forall E \subseteq \mathbb{R}^d$

(iii) More generally, for all $E \subseteq \mathbb{R}^d$, $\exists\, s_E \geq 0$ s.t.

$$\mathcal{H}^k(E) = \begin{cases} +\infty & \text{for } k < s_E \\ 0 & k > s_E \end{cases}$$

s_E is known as the <u>Hausdorff dimension</u> of E.



(iv) $\mathcal{H}^d\big|_{\mathcal{B}_{\mathbb{R}^d}} = \lambda^d\big|_{\mathcal{B}_{\mathbb{R}^d}}$

Measures

σ-algebra, closed under complements and countable unions

Measurable space $(X, \mathcal{M})$

"positive"

Def: A <u>measure</u> μ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, +\infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) μ is countably additive.

Thm (Carathéodory): Given an outer measure μ^*, define

$$\mathcal{M}_{\mu^*} = \{ A \subseteq X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \ \forall E \subseteq X \}$$

Then $\mathcal{M}_{\mu^*}$ is a σ-algebra and $\mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure.

Thm: Suppose (X,d) is a metric space and μ^* is an outer measure on X.

Then $\mu^*|_{\mathbb{B}(X)}$ is a measure iff

$\forall E,F \subseteq X$, $\mathrm{dist}(E,F)>0$,
$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$.

"μ^* is a metric outer measure"

Prop: Fix $d \in \mathbb{N}$, $k \in (0,d)$. $\mathcal{H}^k$ is a Borel measure on $\mathbb{R}^d$.