Lecture 3

260R, Advanced Measure Theory

Reminder: Presentation Topics (Apr ~~16~~ 23)

Makeup Lecture: Friday, April 10th
2-3:45pm
HSSB 3202

Signed Measures (Folland Ch 3)

Def: A signed measure μ on $(X, \mathcal{M})$ is a function $\mu: \mathcal{M} \to [-\infty, +\infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) μ assumes at most one of the values $\pm\infty$

(iii) for $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ disjoint,

$$\nu\left(\bigcup_j E_j\right) = \sum_j \nu(E_j),$$

where if the LHS is finite, the RHS must converge absolutely.

Like positive measures, signed measures satisfy <u>continuity from below and above</u>

<u>Prop:</u> Fix a signed measure ν and $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$.

(i) If $E_1 \subseteq E_2 \subseteq \ldots$, $\nu\left(\bigcup_j E_j\right) = \lim_{j\to\infty} \nu(E_j)$

(ii) If $E_1 \supseteq E_2 \supseteq \ldots$ and $\nu(E_1) \in \mathbb{R}$,

$$\nu\left(\bigcap_j E_j\right) = \lim_{j\to\infty} \nu(E_j).$$

Def: Given a signed measure ν, $A \in \mathcal{M}$, we call the set

(i) positive if $\nu(B) \geq 0 \ \forall \ B \subseteq A$

(ii) negative if $\nu(B) \leq 0 \ \forall \ B \subseteq A$

(iii) null if $\nu(B) = 0. \ \forall \ B \subseteq A$

Warning: If ν is a positive measure, then $A \in \mathcal{M}$ is null $\iff \nu(A) = 0$.

OTOH $d\nu := 1_{[0,1]} d\mathcal{L}^1 - 1_{[-1,0]} d\mathcal{L}^1$ is a signed measure,

$$\nu(E) := \int_E 1_{[0,1]} d\mathcal{L}^1 - \int_E 1_{[-1,0]} d\mathcal{L}^1$$

$$= \mathcal{L}^1(E \cap [0,1]) - \mathcal{L}^1(E \cap [-1,0])$$

We see that $\nu([-1,1]) = 1 - 1 = 0$, but $[-1,1]$ is not a null set, since $[0,1] \subseteq [-1,1]$ and $\nu([0,1]) = 1$.

<u>Lemma</u>: Any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive.

<u>Thm</u>: (Hahn Decomposition) If ν is a signed measure on $(X, \mathcal{A})$, $\exists$ positive set P and negative set N s.t. $P \cup N = X$ and $P \cap N = \emptyset$.
Furthermore, for any other such pair P', N', we have $P \triangle P' = N \triangle N'$ is null.

aka "ν is singular with respect to μ"

Def: Two signed measures μ and ν on $(X, \mathcal{M})$ are mutually singular if $\exists\, E, F \in \mathcal{M}$ s.t. $E \cup F = X$, $E \cap F = \emptyset$, E is ν-null, and F is μ-null. This is denoted $\mu \perp \nu$.

"μ and ν live on disjoint sets"

Ex: $d\mu = 1_{[0,1]}\, d\lambda^1$, $d\nu = 1_{[-2,-1]}\, d\lambda^1$

Thm (Jordan Decomposition): Given a signed measure ν, $\exists!$ positive measures ν^+ and ν^- s.t.

$\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

ν^+ and ν^- are the positive and negative variations of ν

the total variation of ν is the measure $|\nu| := \nu^+ + \nu^-$.

Rmk: Given the Jordan decomposition of a signed measure $\nu = \nu^+ - \nu^-$,

For any set A that is ν-null, we must have $\nu^+(A) = \nu^-(A) = 0$.

Since $\nu^+ \perp \nu^-$ implies $\exists$ partition $X = E \cup F$ with E ν^--null and F-ν^+-null.

Then,

$\nu = \nu^+ - \nu^-$ and E is ν-null

$$\nu^+(A) = \nu^+(A \cap E) = \nu(A \cap E) = 0$$

$$\nu^-(A) = \nu^-(A \cap F) = \nu(A \cap F) = 0$$

If both are >0, then it would contradict that E and F are disjoint.

Rmk: Given a signed measure ν and a positive measure μ. Then,

$$\nu \perp \mu \iff |\nu| \perp \mu$$
$$\iff \nu^+ \perp \mu \text{ and } \nu^- \perp \mu$$

Let's prove $\nu \perp \mu \Rightarrow \nu^+ \perp \mu$ and $\nu^- \perp \mu$.
If $\nu \perp \mu$, then $\exists\ E, F \in \mathcal{M}$ s.t. $E \cup F = X$, $E \cap F = \emptyset$ s.t. E is μ-null and F is ν-null. Write $\nu = \nu^+ - \nu^-$. By previous remark, F is ν^+-null and ν^--null. Thus $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Def: Given a signed measure ν, we can define
$L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-)$.
For any $f \in L^1(\nu)$,

$$\int f \, d\nu := \int f \, d\nu^+ - \int f \, d\nu^-.$$

Lebesgue-Radon-Nikodym Thm

Def: Given a signed measure ν and a positive measure μ on $(X, \mathcal{M})$, ν is absolutely continuous with respect to μ, denote $\nu \ll \mu$, if

$$\forall E \in \mathcal{M}, \mu(E) \Rightarrow \nu(E) = 0.$$

Rmk: $\nu \ll \mu \iff |\nu| \ll \mu$
$\iff \nu^+ \ll \mu$ and $\nu^- \ll \mu$

Exercise :)

Thm: Given a finite signed measure ν and a positive measure μ on $(X, \mathcal{M})$, $\nu \ll \mu \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\mu(E) < \delta$ ensures $|\nu(E)| < \varepsilon$. (*)

Pf:
First, we show "$\Leftarrow$". Fix $E \in \mathcal{M}$ s.t. $\mu(E) = 0$. Fix $\varepsilon > 0$ arbitrary. Since $\mu(E) < \delta$ for any $\delta > 0$, (*) ensures $|\nu(E)| < \varepsilon$.
Thus $\nu(E) = 0$.

Now we show "$\Rightarrow$".

Note that, once we have proved the result for a positive measure ν, it must hold for a signed measure, since

ω positive measure

$\omega \ll \mu \overset{?}{\Leftrightarrow} |\omega| \ll \mu \Leftrightarrow (*)$ holds for $|\omega| \Rightarrow (*)$ holds for ω

$|\omega(E)| = |\omega^+(E) - \omega^-(E)| \leq |\omega|(E)$.

Assume ν is a positive measure. Suppose $(*)$ fails, and we will $\nu \not\ll \mu$. There exists $\varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$ $\exists E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon$.

Define $F_k := \bigcup_{n=k}^{\infty} E_n$ and $F = \bigcap_{k=1}^{\infty} F_k$.

$\quad // \quad (\limsup E_n)$

Then $\mu(F_k) \leq \sum_{n=k}^{\infty} 2^{-n} = 2^{1-k}$,

$\mu(F) = \lim_{k\to\infty} \mu(F_k) = 0$.

OTOH, $\nu(F_k) \geq \varepsilon \ \forall k \in \mathbb{N}$

and since ν finite,

$\nu(F) = \lim_k \nu(F_k) \geq \varepsilon$.

Thus $\nu \not\ll \mu$. $\square$

Q: Does the theorem still hold for ν σ-finite?

Rmk: Given a positive measure μ on $(X, \mathcal{M})$ and $f: X \to [-\infty, +\infty]$ for which either $\int f_+ d\mu < +\infty$ or $\int f_- d\mu < +\infty$, the signed measure

$$\nu(E) := \int_E f \, d\mu$$

satisfies $\nu \ll \mu$.

Abbreviate such measures by

$$d\nu = f \, d\mu.$$