Lecture 4

260R, Advanced Measure Theory

Reminder: Presentation Topics (Apr ~~16~~ 23)
Makeup Lecture: Friday, April 10^{th}
2-3:45pm
HSSB 3202

Lebesgue-Radon-Nikodym Thm

<u>Def</u>: Given a signed measure ν and a positive measure μ on $(X, \mathcal{M})$, ν is <u>absolutely continuous with respect to μ</u>, denote $\nu \ll \mu$, if

$$\forall E \in \mathcal{M}, \mu(E) = 0 \Rightarrow \nu(E) = 0.$$

Rmk: $\nu \ll \mu \iff |\nu| \ll \mu$
$\iff \nu^+ \ll \mu$ and $\nu^- \ll \mu$

Thm: Given a finite signed measure ν and a positive measure μ on $(X, \mathcal{M})$,
$\nu \ll \mu \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t.
$\mu(E) < \delta$ ensures $|\nu(E)| < \varepsilon$. $(*)$

Rmk: The finiteness hypothesis on ν cannot be weakened to σ-finiteness. For example, $\mu = \mathcal{L}^1$ and

$$\nu(E) := \int_E |x| \, d\mu(x)$$

Then ν is σ-finite, $\nu \ll \mu$,

To see (*) fails, take $\varepsilon = 1$.
Suppose $\exists\ \delta > 0$ s.t.
$\mu(E) < \delta$ ensures $|\nu(E)| < 1$.
Then we must have
$|\nu(E)| < 1$ on any interval
of length $< \delta$.

But $\nu(E)$ can become arbitrarily
large as the interval E
moves away from the origin.

To build up to Radon Nikodym
theorem, we require four
lemmas.

First, two 201a reminders...

<u>Lemma 1</u>: Given σ-finite positive measures $\{\nu_i\}_{i=1}^n$ on $(X, \mathcal{M})$, $\exists \{A_k\}_{k=1}^\infty \subseteq \mathcal{M}$ disjoint s.t. $\bigcup_k A_k = X$ and $\nu_i(A_k) < +\infty$ $\forall i, k$.

<u>Pf</u>: If $n=1$, by definition of σ-finiteness, $\exists \{B_k\}_{k=1}^\infty$ s.t. $\bigcup_k B_k = X$, $\nu_1(B_k) < +\infty$.

Define $A_1 := B_1$, $A_k := B_k \setminus \bigcup_{j=1}^{k-1} B_j$.

Suppose the result holds for $\nu_1, \dots, \nu_n$, with partition $\{A_k\}_{k=1}^\infty$. We will show $\exists$ partition for $\nu_1, \dots, \nu_{n+1}$.

By σ-finiteness of ν_{n+1},
$\exists \{B_j\}_{j=1}^{\infty}$ disjoint, $\bigcup_j B_j = X$,
$\nu_{n+1}(B_j) < +\infty$.

Define $C_{j,k} := A_k \cap B_j$. $\square$

<u>Lemma 2:</u> Consider a positive measure μ on $(X, \mathcal{M})$ and suppose $f, g \in L^1(\mu)$. Then

$$\int_E f \, d\mu = \int_E g \, d\mu \quad \forall E \in \mathcal{M} \iff f = g \quad \mu\text{-a.e.}$$

Now, two more lemmas about signed measures.

<u>Lemma:</u> Given a sequence of positive measures $\{\nu_j\}_{j=1}^{\infty}$

and a positive measure μ, $\nu_j \perp \mu \ \forall j \Rightarrow \sum_{j=1}^{\infty} \nu_j \perp \mu$.

Pf: Exercise.

Lemma 4: Given finite positive measures μ and ν on $(X, \mathcal{M})$ either

(i) $\nu \perp \mu$

or

(ii) $\exists\ \varepsilon > 0$ and $E \in \mathcal{M}$ s.t. $\mu(E) > 0$ and $\boxed{\nu \geq \varepsilon\mu \text{ on } E}$.

that is, E is a positive set for $\nu - \varepsilon\mu$.

Pf: Let $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$.

Define $P := \bigcup_{n=1}^{\infty} P_n$, $N := \bigcap_{n=1}^{\infty} N_n = P^c$.

μ is a finite measure

For all $n \in \mathbb{N}$,
$(\nu - \frac{1}{n}\mu)(N) \leq 0 \iff 0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$.
Thus $\nu(N) = 0$.

If $\mu(P) = 0$, then $\mu \perp \nu$.

If $\mu(P) > 0$, then $\mu(P_n) > 0$ for some $n \in \mathbb{N}$. Since P_n is a positive set for $\nu - \frac{1}{n}\mu$, part (ii) holds with $\varepsilon = \frac{1}{n}$. $\square$

Thm (Radon-Nikodym):
Consider σ-finite signed measure ν and σ-finite positive measure μ on $(X, \mathcal{M})$.

There exists
- a σ-finite signed measure λ on $(X, \mathcal{M})$
- a measurable function $f: X \to [-\infty, \infty]$ s.t. $E \mapsto \int_E f \, d\mu$ is σ-finite

"Lebesgue Decomposition"

for which $\boxed{d\nu = d\lambda + f \, d\mu}$
and $\lambda \perp \mu$.

Furthermore, λ is unique and f is unique, up to μ-a.e. equivalence.

Rmk: When $\nu \ll \mu$, then $\lambda = 0$ and $\nu = f\,d\mu$.

The function f is known as the Radon-Nikodym derivative of ν with respect to μ, denoted $\frac{d\nu}{d\mu}$.

Motivation for notation:
Lebesgue decomposition becomes

$$d\nu = \left(\frac{d\nu}{d\mu}\right) d\mu.$$

Pf of Thm:

First, we will show existence of Lebesgue decomposition.

$\boxed{\text{Case I}}$: Suppose ν and μ are finite, positive measures.

Define

$$\mathcal{F} := \{ f : X \to [0,+\infty] \text{ measurable} : \int_E f \, d\mu \leq \nu(E) \ \forall E \in \mathcal{M} \}$$

Note that:

- $\mathcal{F}$ is nonempty, $0 \in \mathcal{F}$
- $f, g \in \mathcal{F} \Rightarrow f \vee g \in \mathcal{F}$

$(f \vee g)(x) = \max(f(x), g(x))$

$$\int_E f \vee g \, d\mu = \int_{E \cap \{f > g\}} f \vee g \, d\mu + \int_{E \cap \{f \leq g\}} f \vee g \, d\mu$$

$$= \int_{E \cap \{f > g\}} f \, d\mu + \int_{E \cap \{f \leq g\}} g \, d\mu$$

$$\leq \nu(E \cap \{f > g\}) + \nu(E \cap \{f \leq g\})$$

$\{x \in X : f(x) > g(x)\}$

$$= \nu(E)$$

- $\{h_n\}_{n=1}^{\infty} \subseteq \mathcal{F}_1$, $h_n \nearrow h \Rightarrow h \in \mathcal{F}_1$

By MCT,
$$\int_E h \, d\mu = \lim_{n\to\infty} \int_E h_n \, d\mu \leq \nu(E).$$

Define $a := \sup_{f \in \mathcal{F}_1} \int_X f \, d\mu \leq \nu(X) < +\infty$.

Let $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{F}_1$ be a maximizing sequence.

$\max_{i=1,\dots,n} f_i(x)$

Define $g_n := \bigvee_{i=1}^{n} f_i \in \mathcal{F}_1$.
Also $\int g_n \, d\mu \geq \int f_n \, d\mu$.

Define $f = \sup_{i \in \mathbb{N}} f_i$. Then

$g_n \nearrow f$ pointwise, so $f \in \mathcal{F}_1$.

Thus, by MCT,

$f \in \mathcal{F}$

$$a \geq \int f \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu$$
$$\geq \lim_{n \to \infty} \int f_n \, d\mu = a.$$

Define $d\lambda = d\nu - f \, d\mu$.
Since $f \in \mathcal{F}$, λ is a positive measure.

Assume, for the sake of contradiction, that $\lambda \perp \mu$ fails. By Lemma 4, $\exists$ $E \in \mathcal{M}$ and $\varepsilon > 0$ s.t. $\mu(E) > 0$ and $\lambda \geq \varepsilon \mu$ on E.

Then $\varepsilon \mathbb{1}_E \, d\mu \le d\lambda = d\nu - f d\mu$
$\iff (f + \varepsilon \mathbb{1}_E) d\mu \le d\nu$

This would imply $f + \varepsilon \mathbb{1}_E \in \mathcal{F}$ and

$$\int f + \varepsilon \mathbb{1}_E \, d\mu = a + \varepsilon \mu(E) > a.$$

This contradicts our construction of f. This proves existence of Lebesgue decomposition.

$\boxed{\text{Case II}}$: Now, suppose ν and μ are σ-finite positive measure. By Lemma 1, $\exists$ partition $\{A_k\}_{k=1}^{\infty}$ of X s.t. $\mu(A_k) < \infty$, $\nu(A_k) < \infty$ $\forall k \in \mathbb{N}$.

Defining

$$\nu_K(E) := \nu(E \cap A_K)$$

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we can apply Case I to these finite measures, obtaining

$$d\nu_K = d\lambda_K + f_K d\mu_K$$

$$\lambda_K \perp \mu_K$$

Since $\mu_K(A_K^c) = \nu_K(A_K^c) = 0$, we have $\lambda_K(A_K^c) = 0$.

We may also assume $f_K = 0$ on A_K^c.

Define $\lambda := \sum_K \lambda_K$, $f := \sum f_K$.

Then

$$d\nu = \sum_k d\nu_k = d\lambda + f\,d\mu$$

<u>Claim</u>: $\lambda_k \perp \mu$

By Lemma 3, $\lambda \perp \mu$.

Remains to show σ-finiteness of λ and $f d\mu$. $\square$