# Lecture 5

260R, Advanced Measure Theory



Reminder: Presentation Topics (Apr 23)

Makeup Lecture: Friday, April $10^{th}$
2-3:45 pm
HSSB 3202

**Lemma 1**: Given $\sigma$-finite positive measures $\{\nu_i\}_{i=1}^n$ on $(X, \mathcal{M})$, $\exists \{A_k\}_{k=1}^\infty \subseteq \mathcal{M}$ disjoint s.t. $\bigcup_k A_k = X$ and $\nu_i(A_k) < +\infty$ $\forall i, k$.

**Rmk**: To extend to $\{\nu_i\}_{i=1}^n$ signed, apply lemma to positive and negative variations.

Lemma 2: Consider a positive measure $\mu$ on $(X, \mathcal{M})$ and suppose $f, g \in L^1(\mu)$. Then

$$\int_E f\,d\mu = \int_E g\,d\mu \;\; \forall E \in \mathcal{M} \iff f = g \;\; \mu\text{-a.e.}$$

Lemma 3: Given a sequence of positive measures $\{\nu_j\}_{j=1}^{\infty}$ and a positive measure $\mu$,

$$\nu_j \perp \mu \;\; \forall j \implies \sum_{j=1}^{\infty} \nu_j \perp \mu.$$

Lemma 4: Given finite positive measures $\mu$ and $\nu$ on $(X, \mathcal{M})$ either

(i) $\nu \perp \mu$

or

(ii) $\exists\, \varepsilon > 0$ and $E \in \mathcal{M}$ s.t.

$\mu(E) > 0$ and $\nu \geq \varepsilon \mu$ on $E$.

Thm (Radon-Nikodym):
Consider $\sigma$-finite signed measure $\nu$ and $\sigma$-finite positive measure $\mu$ on $(X, \mathcal{M})$.

There exists
- a $\sigma$-finite signed measure $\lambda$ on $(X, \mathcal{M})$
- a measureable function $f: X \to [-\infty, \infty]$ s.t. $E \mapsto \int_E f \, d\mu$ is $\sigma$-finite

"Lebesgue Decomposition"

for which $\boxed{d\nu = d\lambda + f\,d\mu}$
and $\lambda \perp \mu$.

Furthermore, $\lambda$ is unique and $f$ is unique, up to $\mu$-a.e. equivalence.

Rmk: When $\nu << \mu$, then

$\lambda \perp \mu$

$X = E \cup F$ $\quad d\nu = \underset{E}{d\lambda} + \overbrace{f\,d\mu}^{F}$

$\nu(A) = \lambda(A) + \int_A f\,d\mu$, $A \subseteq E$

$0 = \lambda(A) + 0$

Since $F = E^c$ is $\lambda$-null, we see that...

$\lambda = 0$ and $d\nu = f\,d\mu$.

The function $f$ is known as the Radon-Nikodym derivative of $\nu$ with respect to $\mu$, denoted $\frac{d\nu}{d\mu}$.

Pf of Thm:
Last time, we were proving existence of the Lebesgue decomposition for $\nu$ and $\mu$ $\sigma$-finite positive measures.

$\boxed{\text{Case II}}$: Now, suppose $\nu$ and $\mu$ are $\sigma$-finite positive measures. By Lemma 1, $\exists$ partition $\{A_k\}_{k=1}^{\infty}$ of $X$ s.t. $\mu(A_k) < \infty$, $\nu(A_k) < \infty$ $\forall k \in \mathbb{N}$.

Defining

$$\nu_k(E) := \nu(E \cap A_k)$$
$$\mu_k(E) := \mu(E \cap A_k)$$

we can apply Case I to these finite measures, obtaining

$$d\nu_k = d\lambda_k + f_k\, d\mu_k$$
$$\lambda_k \perp \mu_k$$

Since $\mu_k(A_k^c) = \nu_k(A_k^c) = 0$, $(*)$
$\lambda_k(A_k^c) = 0 \ \forall \ k$. Also, we may assume $f_k = 0$ on $A_k^c$.

Define $\lambda := \sum_k \lambda_k$, $f := \sum f_k$.

$\sigma$-finite by defn

Then

$$d\nu = \sum_k d\nu_k = d\lambda + \underbrace{f\,d\mu}$$

Note that $\lambda_k \perp \mu$, $\forall k \in \mathbb{N}$. To see this, note that since $\lambda_k \perp \mu_k$, $\exists$ partition $E_k, F_k$, where $E_k$ is $\mu_k$-null, $F_k$ is $\lambda_k$-null.

Then $\mu(E_k \cap A_k) = \mu_k(E_k) = 0$
and $\lambda_k((E_k \cap A_k)^c) = \lambda_k(F_k \cup A_k^c)$
$\leq \lambda_k(F_k) + \lambda_k(A_k^c) = 0$.

Thus $E_k \cap A_k$ and its complement partition $X$ into $\mu$-null and $\lambda_k$ null sets, showing $\lambda_k \perp \mu$.

By Lemma 3, $\lambda = \sum_{k=1}^{\infty} \lambda_k$ satisfy $\lambda \perp \mu$.

$\boxed{\text{Case 3}}$: Finally, suppose that $\nu$ is a $\sigma$-finite signed measure. By Case 2,

$$d\nu^+ = d\lambda^{(1)} + f^{(1)} d\mu, \quad \lambda^{(1)} \perp \mu$$
$$d\nu^- = d\lambda^{(2)} + f^{(2)} d\mu, \quad \lambda^{(2)} \perp \mu$$

Since $\nu$ is a signed measure, it can't attain both $\pm$-$\infty$. WLOG, assume $\nu$ is never $-\infty$, so

$$+\infty > \nu^-(X) = \lambda^{(2)}(X) + \int_X f^{(2)} d\mu.$$

Thus, we have

$$d\nu = d\nu^+ - d\nu^-$$
$$= d\underbrace{(\lambda^{(1)} - \lambda^{(2)})}_{=:\lambda} + \underbrace{(f^{(1)} - f^{(2)})}_{=:f} d\mu$$

Both $\lambda$ and $f d\mu$ $\sigma$-finite. Finally, since $\lambda^{(1)} \perp \mu$ and $\lambda^{(2)} \perp \mu$, Lemma 3 ensures

$$\overbrace{\lambda^{(1)} + \lambda^{(2)}}^{|\lambda|} \perp \mu$$

so $\lambda \perp \mu$.

Uniqueness

Suppose we also have

$$d\nu = d\lambda' + f' d\mu, \quad \lambda' \perp \mu.$$

By $\sigma$-finiteness and Lemma 1, $\exists \{A_k\}_{k=1}^{\infty}$ disjoint partition s.t. $A_k$ is finite for all measures in this problem :)

Then, $\forall B \in \mathcal{M}$,

$$\nu(B \cap A_k) = \lambda'(B \cap A_k) + \int_{A_k \cap B} f' d\mu$$

$$= \lambda(B \cap A_k) + \int_{A_k \cap B} f \, d\mu.$$

Thus,

$$(\lambda-\lambda')(B\cap A_k)=\int_{B\cap A_k}(f-f')\,d\mu. \quad (*)$$

If $\mu(B)=0$, $(\lambda-\lambda')(B\cap A_k)=0$ $\forall k\in\mathbb{N} \Rightarrow (\lambda-\lambda')(B)=0$.

Since $\mu\perp\lambda$ and $\mu\perp\lambda'$ $\Leftrightarrow \mu\perp|\lambda|$ and $\mu\perp|\lambda'|$ $\xrightarrow{\text{Lemma 3}} \mu\perp|\lambda|+|\lambda'|$.

Let $E,F$ be corresponding partition $E$ $|\lambda|+|\lambda'|$-null, $F$ $\mu$-null.

Therefore, for all $B \in \mathcal{M}$,

$$\begin{aligned}
\lambda(B) &= \lambda(B \cap E) + \lambda(B \cap F) \\
&= 0 + \lambda(B \cap F) \\
&= 0 + \lambda'(B \cap F) \quad \mu(B \cap F) = 0 \\
&= \lambda'(B \cap E) + \lambda'(B \cap F) \\
&= \lambda'(B)
\end{aligned}$$

$\Rightarrow \lambda = \lambda'$.

Since LHS of $(*)$ is always zero, by Lemma 2,

$$f \mathbb{1}_{A_K} = f' \mathbb{1}_{A_K} \quad \mu\text{-a.e.}$$

Thus $f = f'$ $\mu$-a.e. $\square$

This uniqueness result gives some nice basic properties of the Radon Nikodym derivative.

Also suppose $\nu_1+\nu_2$ well-defined

**Lemma**: Suppose $\nu_1$ and $\nu_2$ are $\sigma$-finite signed measures, $\mu$ $\sigma$-finite positive measure and $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. Then
$$\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}, \quad \mu\text{-a.e.}$$

Pf: By the previous theorem
$$d\nu_1 = \frac{d\nu_1}{d\mu}\, d\mu, \quad d\nu_2 = \frac{d\nu_2}{d\mu}\, d\mu$$

Thus, $d(\nu_1+\nu_2) = \left(\frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}\right) d\mu.$

Uniqueness of the Lebesgue decomposition for $\nu_1 + \nu_2$ gives the result.

**Prop:** Suppose $\nu$ is a $\sigma$-finite signed measure and $\mu, \lambda$ are $\sigma$-finite positive measures s.t. $\nu \ll \mu$ and $\mu \ll \lambda$. Then $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}, \ \lambda\text{-a.e.}$$

**Pf:** By Lebesgue decomposition,

$$d\nu = \frac{d\nu}{d\lambda} d\lambda$$

$$d\nu = \frac{d\nu}{d\mu} d\mu = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} d\lambda.$$

Uniqueness again gives the result.

Ex: Suppose $\mu = \mathcal{L}^d$, $\nu = \delta_{\{0\}}$. Then $\mu \perp \nu$. The Lebesgue decomposition is

$$d\nu = \underbrace{d\nu} + \underbrace{0}\, d\mu$$

We do not have $\nu \ll \mu$ and $\overline{\dfrac{d\nu}{d\mu}}$ is not defined.

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## Lebesgue-Besicovitch Differentiation

Specialize to $(X, \mathcal{M}) = (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$

By Radon-Nikodym theorem, given $f \in L^1(\mathcal{L}^d)$, then

(*) $d\nu = f \, d\mathcal{L}^d$

is a signed measure and $\frac{d\nu}{d\mathcal{L}^d} = f$.

Given $\nu$ of the form (*), there is another way we could "find" the $f$:

$$\frac{\nu(B_r(x))}{\mathcal{L}^d(B_r(x))} = \underbrace{\frac{1}{\mathcal{L}^d(B_r(x))} \int_{B_r(x)} f(y) \, d\mathcal{L}^d(y)}_{\text{average value of 'f' over } B_r(x)}$$

morally...

$$\xrightarrow[r \to 0]{} f(x)$$

(modulo any issues that $f$ is $\mathcal{L}^d$-a.e. defined)

This alternative characterization of the Radon-Nikodym derivative will allow us to study weak notions of differentiability and prove Fundamental Theorem of Calculus.

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A key tool...

Thm (Besicovitch's covering Theorem): For $d \in \mathbb{N}$, $\exists\ \xi(d) \in \mathbb{N}$ with the following property...

Let $\mathcal{F}$ be a family of closed, nondegenerate balls in $\mathbb{R}^d$. Let $C$ denote the centers of the balls.

Assume that either...

(i) $C \subseteq \mathbb{R}^d$ is bounded

(ii) $\sup\{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F}\} < +\infty$

Then $\exists\ \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{\xi(d)}$ (possibly empty) subfamilies of $\mathcal{F}$ s.t.

(a) Each $\mathcal{F}_i$ is disjoint and countable

(b) $C \subseteq \displaystyle\bigcup_{i=1}^{\xi(d)} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B}_i$

Q: If (i) holds and (ii) fails, isn't result trivially true by taking an arbitrarily large ball.