Lecture 6

260R, Advanced Measure Theory

Reminder: Presentation Topics (Apr 23)

make sure to come to office hours to show me detailed notes ahead of time

Goal: ≤ 15 minutes (≤ 3 pages notes)

Next week: sign up for topics for second round of presentations

Thm (Radon-Nikodym):
Consider σ-finite signed measure ν and σ-finite positive measure μ on $(X, \mathcal{M})$.

There exists
- a σ-finite signed measure λ on $(X, \mathcal{M})$
- a measurable function $f: X \to [-\infty, \infty]$ s.t. $E \mapsto \int_E f \, d\mu$ is σ-finite

"Lebesgue Decomposition"

for which $\boxed{d\nu = d\lambda + f\, d\mu}$
and $\lambda \perp \mu$.

Furthermore, λ is unique and f is unique, up to μ-a.e. equivalence.

This uniqueness result gives some nice basic properties of the Radon Nikodym derivative.

<u>Lemma</u>: Suppose ν_1 and ν_2 are σ-finite signed measures, Also suppose $\nu_1 + \nu_2$ well-defined, μ σ-finite positive measure and $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$.

Then
$$\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}, \quad \mu\text{-a.e.}$$

Prop: Suppose ν is a σ-finite signed measure and μ, λ are σ-finite positive measures s.t. $\nu \ll \mu$ and $\mu \ll \lambda$. Then $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}, \quad \lambda\text{-a.e.}$$

Lebesgue-Besicovitch Differentiation

Specialize to $(X, \mathcal{M}) = (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$

Thm (Besicovitch's covering Theorem): For $d \in \mathbb{N}$, $\exists\ \xi(d) \in \mathbb{N}$ with the following property...

Let $\mathcal{F}$ be a family of closed, nondegenerate balls in $\mathbb{R}^d$. Let C denote the centers of the balls.

Assume that either...

(i) $C \subseteq \mathbb{R}^d$ is bounded

(ii) $\sup\{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F}\} < +\infty$

Then $\exists\ \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{\xi(d)}$ (possibly empty) subfamilies of $\mathcal{F}$ s.t.

(a) Each $\mathcal{F}_i$ is disjoint and countable

that is the closed balls in $\mathcal{F}_i$ are mutually disjoint

(b) $C \subseteq \displaystyle\bigcup_{i=1}^{\xi(d)} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B}$

Q: If (i) holds and (ii) fails, isn't result trivially true by taking an arbitrarily large ball?

A: Yes. If C is bounded, $\mathrm{diam}(C) < +\infty$ so if we choose $\overline{B} \in \widetilde{\mathcal{F}}$ so that $\mathrm{diam}(\overline{B}) > 2\,\mathrm{diam}(C)$, then $C \subseteq \overline{B}$.

<u>Rmk</u>: The theorem is formulated in this way since, in many applications, C is clearly bounded... then don't need to check anything about diameters.

We will always assume μ is a locally finite Borel measure on $\mathbb{R}^d$.

μ gives finite measure to all balls

In our study of Radon measures, we will prove that all such measures satisfy the following:

$\forall E \in \mathcal{B}_{\mathbb{R}^d}$,

inner regular: $\mu(E) = \inf\{\mu(U) : U \text{ open}, U \supseteq E\}$

outer regular: $\mu(E) = \sup\{\mu(K) : K \text{ cpt}, K \subseteq E\}$

continuous functions are dense in $L^1(\mu)$: $\forall f \in L^1(\mu)$, $\forall \varepsilon > 0$, $\exists g \in C(\mathbb{R}^d)$ s.t. $\|f-g\|_{L^1(\mu)} < \varepsilon$.

Def: A measureable function f is **locally integrable** w.r.t. μ, denoted $f \in L^1_{loc}(\mu)$, if

$$\int_K |f(x)|\, d\mu(x) < +\infty, \quad \forall K \in \mathcal{B}_{\mathbb{R}} \text{ bounded}$$

Def: Given $f \in L^1_{loc}(\mu)$, $r > 0$, $x \in \mathbb{R}^d$, the average value of f on $\overline{B_r(x)}$ is

$$A_r f(x) := \begin{cases} \dfrac{1}{\mu(\overline{B_r(x)})} \displaystyle\int_{\overline{B_r(x)}} f(y)\, d\mu(y) & \text{otherwise} \\ 0 & \text{if } \mu(\overline{B_r(x)}) = 0 \end{cases}$$

Rmk: If $f \in C(\mathbb{R}^d) \subseteq L^1_{loc}(\mu)$, we have that, for any $x \in \mathbb{R}^d$ s.t. $\mu(\overline{B_r(x)}) > 0 \ \forall r > 0$, $(*)$ we have

$$A_r f(x) \xrightarrow{r \to 0} f(x).$$

To see why this is true, note that $\forall \varepsilon > 0$, $\exists R$ s.t. $r \leq R$, $|x-y| \leq r \Rightarrow |f(x)-f(y)| < \varepsilon$.

Thus

$$|A_r f(x) - f(x)|$$

$$= \left| \frac{1}{\mu(\overline{B_r(x)})} \int_{\overline{B_r(x)}} f(y)\, d\mu(y) - f(x) \cdot \underbrace{\frac{1}{\mu(\overline{B_r(x)})} \int_{\overline{B_r(x)}} d\mu(y)}_{=1} \right|$$

$$= \left| \frac{1}{\mu(\overline{B_r(x)})} \int_{\overline{B_r(x)}} f(y) - f(x)\, d\mu(y) \right|$$

$$\leq \frac{1}{\mu(\overline{B_r(x)})} \int_{\overline{B_r(x)}} \underbrace{|f(y)-f(x)|}_{<\varepsilon} \, d\mu(y)$$

$$< \varepsilon$$ provided $r \leq R$.

This shows $\lim_{r\to 0} A_r f(x) = f(x)$.

Rmk: $(\dagger)$ holds for μ-a.e. x.
That is, the set
$S := \{x \in \mathbb{R}^d : \mu(\overline{B_r(x)}) = 0 \text{ for some } r_x > 0\}$
is μ-null. For any such x,
$\exists\, q_x \in \mathbb{Q} \cap \mathbb{R}_{>0}$, $p_x \in \mathbb{Q}^d$ s.t.
$x \in B_{q_x}(p_x) \subseteq \overline{B_{r_x}(x)}$, so $\mu(B_{q_x}(p_x)) = 0$.

Let $\mathcal{F}_1 = \{B_{q_x}(p_x) : x \in S\}$

$$\mu(S) \leq \sum_{B \in \mathcal{F}_1} \mu(B) \leq \sum_{B\in\mathcal{F}_1} 0 = 0.$$

Major goal of this week:
extend $A_r f(x) \xrightarrow{r\to 0} f(x)$
all $f \in L^1_{loc}(\mu)$.

<u>Lemma</u>: For all $r > 0$,

$x \mapsto \mu(B_r(x))$ is lsc,
$x \mapsto \mu(\overline{B_r(x)})$ is usc.

<u>Pf</u>: Exercise

<u>Lemma</u>: If $f \in L^1_{loc}(\mu)$, $r > 0$,
$A_r f(x)$ is measureable in x.
Furthermore, if x, r are such that $A_r f(x) > 0$, then $r_n \searrow r$,
$\lim_{n\to\infty} A_{r_n} f(x) = A_r f(x)$.

Pf: Applying the previous lemma to μ and $|f|d\mu$ we see that

$$x \mapsto \mu(\overline{B_r(x)}) + 1_{\underbrace{\{x : \mu(\overline{B_r(x)})=0\}}_{\text{Borel set}}}(x)$$

$$x \mapsto \int_{\overline{B_r(x)}} f_+(y)\,d\mu(y)$$

$$x \mapsto \int_{\overline{B_r(x)}} f_-(y)\,d\mu(y)$$

are all measureable.

Therefore, $\forall r > 0$,

$$A_r f(x) = \begin{cases} \dfrac{1}{\mu(\overline{B_r(x)})} \displaystyle\int_{\overline{B_r(x)}} f(y)\,d\mu(y) & \text{else,} \\ 0 & \text{if } \mu(\overline{B_r(x)}) = 0. \end{cases}$$

is measureable in x.

Suppose $x \in \mathbb{R}^d$, $r > 0$ satisfy $A_r f(x) > 0$. Then $\mu(\overline{B_r(x)}) > 0$.

Fix $r_n \searrow r$. By cty from above, for any (locally finite ν,

$$\lim_{n\to\infty} \nu(\overline{B_{r_n}(x)}) = \nu\left(\bigcap_{n\in\mathbb{N}} \overline{B_{r_n}(x)}\right)$$

$$= \nu(\overline{B_r(x)})$$

$|y-x| \le r_n \ \forall n \in \mathbb{N} \iff |y-x| \le r$

Applying to $\nu = \mu$ and $\nu = f d\mu$,

$$\frac{1}{\mu(\overline{B_r(x)})} \quad \text{and} \quad \int_{\overline{B_r(x)}} f(y)\, d\mu(y)$$

are both right continuous.

Thus, their product is right cts. $\square$

Def: Given $f \in L^1_{loc}(\mu)$, the Hardy-Littlewood maximal function $Hf : \mathbb{R}^d \to [0, +\infty]$ is

$$Hf(x) := \sup_{r>0} A_r|f|(x).$$

Rmk: For any $a > 0$,

$$Hf^{-1}((a,+\infty)) = \bigcup_{r>0} \overbrace{(A_r|f|)^{-1}((a,+\infty))}^{\in \mathcal{B}_{\mathbb{R}^d}}$$

$$x \in Hf^{-1}((a,+\infty)) = \bigcup_{q>0,\, q\in\mathbb{Q}} \overbrace{(A_q|f|)^{-1}((a,+\infty))}^{\in \mathcal{B}_{\mathbb{R}^d}}$$

$\Longleftrightarrow Hf(x) = \sup_{r>0} A_r|f|(x) > a$

$\Longleftrightarrow$ a is <u>not</u> an upper bound of $\{A_r|f|(x) : r > 0\}$

where the last equality followed by right continuity,

$$A_r Hf(x) > a \Longleftrightarrow \exists\, q \in \mathbb{Q},\ q > 0,$$
$$\text{s.t. } A_q Hf(x) > a$$

Thus $Hf^{-1}((a,+\infty)) \in \mathcal{B}_{\mathbb{R}^d}$ and Hf is measureable.

Thm (Hardy Littlewood maximal theorem): For all $f \in L^1(\mu)$ and $\alpha > 0$

$$\mu(\{x : Hf(x) > \alpha\}) \leq \frac{\xi(d)}{\alpha} \|f\|_{L^1(\mu)}.$$

Rmk: It turns out that $H : L^1(\mu) \not\to L^1(\mu)$. The previous thm show $H : L^1(\mu) \to L^{1,\infty}(\mu)$.

Proof next time :)