Lecture 7

260R, Advanced Measure Theory

Reminder: Presentation Topics (Apr 23)

Thm (Besicovitch's covering Theorem): For $d \in \mathbb{N}$, $\exists\ \xi(d) \in \mathbb{N}$ with the following property...

Let $\mathcal{F}$ be a family of closed, nondegenerate balls in $\mathbb{R}^d$. Let C denote the centers of the balls.

Assume that either...

(i) $C \subseteq \mathbb{R}^d$ is bounded

(ii) $\sup\{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F}\} < +\infty$

Then $\exists\ \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{\xi(d)}$ (possibly empty) subfamilies of $\mathcal{F}$ s.t.

(a) Each $\mathcal{F}_i$ is $\underbrace{\text{disjoint}}_{\text{that is the closed balls in } \mathcal{F}_i \text{ are mutually disjoint}}$ and countable

(b) $C \subseteq \bigcup_{i=1}^{\xi(d)} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B}$

In our study of Lebesgue-Besicovitch differentiation, we will always assume μ is a locally finite Borel measure on $\mathbb{R}^d$.

In our study of Radon measures, we will prove that all such measures satisfy the following:
$\forall E \in \mathcal{B}_{\mathbb{R}^d}$,

outer regular: $\mu(E) = \inf\{\mu(U) : U \text{ open}, U \supseteq E\}$
inner regular: $\mu(E) = \sup\{\mu(K) : K \text{ cpt}, K \subseteq E\}$
continuous functions are dense in $L^1(\mu)$: $\forall f \in L^1(\mu)$, $\forall \varepsilon > 0$, $\exists g \in C(\mathbb{R}^d)$ s.t. $\|f-g\|_{L^1(\mu)} < \varepsilon$.

Furthermore, note that if ν is a locally finite signed Borel measure on $\mathbb{R}^d$, so is $|\nu|$.

Def: A measureable function f is <u>locally integrable</u> <u>w.r.t. μ</u>, denoted $f \in L^1_{loc}(\mu)$, if $\int_K |f(x)| \, d\mu(x) < +\infty$, $\forall K \in \mathcal{B}_{\mathbb{R}}$ bounded

Def: Given $f \in L^1_{loc}(\mu)$, $r > 0$, $x \in \mathbb{R}^d$, <u>the</u> average value of f on $\overline{B_r(x)}$ is

$$A_r f(x) := \begin{cases} \dfrac{1}{\mu(\overline{B_r(x)})} \displaystyle\int_{\overline{B_r(x)}} f(y) \, d\mu(y) & \text{otherwise} \\ 0 & \text{if } \mu(\overline{B_r(x)}) = 0 \end{cases}$$

Rmk: If $f \in C(\mathbb{R}^d) \subseteq L^1_{loc}(\mu)$, we have that, for any $x \in \mathbb{R}^d$ s.t. $\mu(\overline{B_r(x)}) > 0$ $\forall r > 0$,

$$A_r f(x) \xrightarrow{r \to 0} f(x). \quad (*)$$

Rmk: $(*)$ holds for μ-a.e. x. That is, the following set is μ-null:

$$S := \{x \in \mathbb{R}^d : \mu(\overline{B_r(x)}) = 0 \text{ for some } r > 0\}.$$

Major goal of this week: extend $A_r f(x) \xrightarrow{r \to 0} f(x)$ to all $f \in L^1_{loc}(\mu)$.

Lemma: For all $r > 0$, $x \mapsto \mu(B_r(x))$ is lsc, $x \mapsto \mu(\overline{B_r(x)})$ is usc.

<u>Lemma</u>: If $f \in L^1_{loc}(\mu)$, $r>0$, $A_r f(x)$ is measureable in x.
Furthermore, if x, r are such that $A_r f(x) > 0$, then $r_n \searrow r$ implies

$$\lim_{n \to \infty} A_{r_n} f(x) = A_r f(x).$$

<u>Def</u>: Given $f \in L^1_{loc}(\mu)$, the Hardy-Littlewood maximal function $Hf : \mathbb{R}^d \to [0, +\infty]$ is

$$Hf(x) := \sup_{r>0} A_r |f|(x).$$

<u>Rmk</u>: Hf is measureable.

Thm (Hardy Littlewood Maximal Theorem): For all $f \in L^1(\mu)$ and $\alpha > 0$

$$\mu\Big(\overbrace{\{x : Hf(x) > \alpha\}}^{E_\alpha :=}\Big) \le \frac{\xi(d)}{\alpha} \|f\|_{L^1(\mu)}.$$

Pf: Assume, WLOG, $E_\alpha \neq \emptyset$. For all $x \in E_\alpha$, $\exists\, r_x > 0$ s.t. $A_{r_x}|f|(x) > \alpha$, that is,

$$\frac{1}{\alpha} \int_{\overline{B_{r_x}(x)}} |f(y)|\, d\mu(y) > \mu(\overline{B_{r_x}(x)}).$$

Fix $K \subseteq E_\alpha$ compact. WLOG, $K \neq \emptyset$.

Consider the family $\mathcal{F}$ of balls $B_{r_x}(x)$, $x \in K$.

Therefore, by Besicovitch, $\exists \{\mathcal{F}_i\}_{i=1}^{\xi(d)}$ s.t.

$$\mu(K) \leq \sum_{i=1}^{\xi(d)} \sum_{\overline{B} \in \mathcal{F}_i} \mu(\overline{B})$$

$$\leq \underbrace{\sum_{i=1}^{\xi(d)} \sum_{\overline{B} \in \mathcal{F}_i} \frac{1}{\alpha} \int_{\overline{B}} |f(y)| \, d\mu(y)}_{= \frac{1}{\alpha} \int_{\bigcup_{\overline{B} \in \mathcal{F}_i}} |f(y)| \, d\mu(y)}$$

$$\leq \frac{\xi(d)}{\alpha} \|f\|_{L^1(\mu)}$$

The result then follows by inner regularity. $\square$

Now, we will use the maximal theorem to prove...

<u>Thm</u> (Lebesgue Differentiation Theorem): Suppose $f \in L^1_{loc}(\mu)$. Then, for μ-a.e. x,

$$(\star) \quad \lim_{r \to 0} \frac{1}{\mu(E_r)} \int_{\mu(E_r)} |f(y) - f(x)| \, d\mu(y) = 0,$$

for any $\underline{\{E_r\}_{r>0}} \subseteq \mathcal{B}_{\mathbb{R}^d}$ s.t.
(i) $E_r \subseteq \overline{B_r(x)}$ $\forall r > 0$
(ii) $\exists \alpha > 0$ s.t. $\mu(E_r) \geq \alpha \mu(\overline{B_r(x)})$ $\forall r > 0$.

We will say that any $\{E_r\}_{r>0}$ satisfying (i) and (ii) "shrinks nicely" to x.

Rmk: Note that (*) implies

$$\lim_{r\to 0}\left|\frac{1}{\mu(E_r)}\int_{E_r} f(y)-f(x)\,d\mu(y)\right|=0$$

$$\|$$

$$\lim_{r\to 0}\left|\frac{1}{\mu(E_r)}\int_{E_r} f(y)\,d\mu(y)-f(x)\right|$$

Our strategy to prove the theorem is to first prove this weaker property for $E_r=\overline{B_r(x)}$.

Prop: Given $f\in L^1_{loc}(\mu)$, for μ-a.e. x,

$$\lim_{r\to 0} A_r f(x)=f(x).$$

Pf: Fix $N \in \mathbb{N}$. We will show $\lim_{r \to 0} A_r f(x) = f(x)$ for μ-a.e. $x \in B_N(0)$.

μ-null set on which result fails

Then, writing $B_N(0) = G_N \cup Z_N$, disjoint, we see that the result holds everywhere except $\bigcup_N Z_N$, which is μ-null!

For $x \in B_N(0)$ and $r \leq 1$, $A_r f(x)$ only depends on $f(y)$ for $y \in B_{N+1}(0)$. Thus, replacing f with $f \mathbb{1}_{B_{N+1}(0)}$ we may

assume $f \in L^1(\mu)$.

Fix $\varepsilon > 0$. Choose $g \in C(\mathbb{R}^d)$ s.t.

$$\|g - f\|_{L^1(\mu)} < \varepsilon.$$

We know that $A_r g(x) \xrightarrow{r \to 0} g(x)$ for μ-a.e. x. Let T denote the set of such x.

Thus,

$$\limsup_{r \to 0} |A_r f(x) - f(x)|$$

$$= \limsup_{r \to 0} |A_r(f-g)(x) + (A_r g - g)(x) + (g-f)(x)|$$

$$\leq H(f-g)(x) + 0 + |g-f|(x)$$

Define

$$E_\alpha = \{x \in T : \limsup_{r \to 0} |A_r f(x) - f(x)| > \alpha\}$$

$$F_\alpha = \{x \in T : |f-g|(x) > \alpha\}$$

Then

$$E_\alpha \subseteq F_{\frac{\alpha}{2}} \cup \{x \in T : H(f-g)(x) > \tfrac{\alpha}{2}\}.$$

By HL Max Thm,

$$\mu(\{x : H(f-g)(x) > \tfrac{\alpha}{2}\}) \leq \frac{2\,\xi(d)}{\alpha} \overbrace{\|f-g\|_{L^1}}^{<\varepsilon}$$

Similarly,

$$\mu(F_{\frac{\alpha}{2}}) = \int_{F_{\frac{\alpha}{2}}} 1 \, d\mu$$

$$\leq \frac{2}{\alpha} \overbrace{\int_{F_{\frac{\alpha}{2}}} |f-g| \, d\mu}^{<\varepsilon}$$

Thus,

$$\mu(E_\alpha) \le \frac{2\xi(d)\varepsilon}{\alpha} + \frac{2}{\alpha}\varepsilon .$$

Since $\varepsilon > 0$ arb, $\mu(E_\alpha) = 0 \ \forall \alpha > 0$.

Finally, $\mu\left(\bigcup_n E_{\frac{1}{n}}\right) = 0$ and

$x \notin \bigcup_n E_{\frac{1}{n}}$ implies

$$\limsup_{r \to 0} |A_r f(x) - f(x)| = 0.$$

Since $\mu(T^c) = 0$, this gives the result.

Now we can prove the main result.

Pf: We will first show the result for $E_r = \overline{B_r(x)}$.

Fix $a \in \mathbb{R}$. Applying previous prop to $g_a(x) := |f(x) - a|$, $\exists$ E_a μ-null s.t. $\forall x \in E_a^c$,

$$\lim_{r \to 0} \frac{1}{\mu(\overline{B_r(x)})} \int_{\overline{B_r(x)}} |f(y) - a| \, dy = |f(x) - a|.$$

Then $E := \bigcup_{q \in \mathbb{Q}} E_q$ is μ-null and

if $x \in E^c$, $\forall \varepsilon > 0$, $\exists$ $q_x \in \mathbb{Q}$ s.t.

$$|f(x) - q_x| < \varepsilon$$

$$\Rightarrow |f(y) - f(x)| < |f(y) - q_x| + \varepsilon$$

Thus,

$$\limsup_{r \to 0} \frac{1}{\mu(\overline{B_r(x)})} \int_{\overline{B_r(x)}} |f(x) - f(y)|\, dy$$

$$\leq \limsup_{r \to 0} \frac{1}{\mu(\overline{B_r(x)})} \int_{\overline{B_r(x)}} |f(y) - q_x|\, dy + \varepsilon$$

$$\overset{x \in E^c}{=} |f(x) - q_x| + \varepsilon \overset{\text{choice of } q_x.}{\leq} 2\varepsilon$$

Since $\varepsilon > 0$ was arbitrary, this proves the result for $E_r = \overline{B_r(x)}$.

Now, let $\{E_r\}_{r>0}$ be any sets shrinking nicely to x

$$\frac{1}{\mu(E_r)} \int_{E_r} |f(y)-f(x)| \, d\mu(y)$$

$$\leq \frac{1}{\alpha \, \mu(B_r(x))} \int_{B_r(x)} |f(y)-f(x)| \, d\mu(y) \xrightarrow{r \to 0} 0$$

for μ-a.e. x. $\square$

We will now relate the Radon Nikodym derivative to average behavior of measures over balls.

Thm: Let ν and μ be locally finite signed and positive Borel measures on $\mathbb{R}^d$. Let

$$d\nu = d\lambda + f\,d\mu$$

be the Lebesgue decomposition.

Then, for μ-a.e. x,

$$f(x) = \lim_{r \to 0} \frac{\nu(E_r)}{\mu(E_r)}$$

for any family $\{E_r\}_{r>0}$ that shrinks nicely to x.

Pf: Note that $d|\nu| = d|\lambda| + |f|d\mu$, so λ and $fd\mu$ are locally finite, so $f \in L^1_{loc}(\mu)$.

Recall that $\lambda \perp \mu \iff |\lambda| \perp \mu$.
Let $X = E \cup F$ be a partition into μ-null, λ-null set.

By the Lebesgue decomp,

$$\lim_{r \to 0} \frac{\nu(E_r)}{\mu(E_r)} = \lim_{r \to 0} \frac{\lambda(E_r) + \int_{E_r} f d\mu}{\mu(E_r)}$$

$\to f(x)$ μ-a.e.

Thus, it suffices to show $\lim_{r \to 0} \frac{\lambda(E_r)}{\mu(E_r)} = 0$.

Furthermore, since E_r "shrinks nicely" to x, $\exists \alpha > 0$ s.t.

$$\left|\frac{\lambda(E_r)}{\mu(E_r)}\right| \le \frac{|\lambda|(E_r)}{\mu(E_r)} \le \frac{|\lambda|(\overline{B_r(x)})}{\alpha\mu(\overline{B_r(x)})}$$

We will show RHS $\to 0$ for μ-a.e. x.

Define

$$F_k = \left\{x \in F : \limsup_{r \to 0} \frac{|\lambda|(\overline{B_r(x)})}{\mu(\overline{B_r(x)})} > \frac{1}{k}\right\}$$

Claim: $\mu(F_k) = 0 \;\forall k$.

If the claim holds,

so

$$0 = \mu\left(\bigcup_k F_k\right)$$

$$= \mu\left(\left\{x \in F : \limsup_{r \to 0} \frac{|\lambda|(\overline{B_r(x)})}{\mu(\overline{B_r(x)})} > 0\right\}\right)$$

Furthermore $\mu(F^c) = \mu(E) = 0$.

Thus $\limsup_{r \to 0} \frac{|\lambda|(\overline{B_r(x)})}{\mu(\overline{B_r(x)})} = 0$ μ-a.e..

Next time use Besicovitch to prove claim :)