Lecture 8

260R, Advanced Measure Theory
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Reminder: Presentation Topics - Apr 23
- May 5
- May 14

Thm (Lebesgue Differentiation Theorem): Suppose $f \in L^1_{loc}(\mu)$. Then, for μ-a.e. x,

$$(\star) \quad \lim_{r \to 0} \frac{1}{\mu(E_r)} \int_{\mu(E_r)} |f(y) - f(x)| \, d\mu(y) = 0,$$

"shrink nicely"

for any $\{E_r\}_{r>0} \subseteq \mathcal{B}_{\mathbb{R}^d}$ s.t.

(i) $E_r \subseteq \mathbb{B}_r(x) \quad \forall r > 0$

(ii) $\exists \alpha > 0$ s.t. $\mu(E_r) \geq \alpha \mu(\overline{\mathbb{B}_r(x)})$ $\forall r > 0$.

Thm: Let ν and μ be locally finite signed and positive Borel measures on $\mathbb{R}^d$. Let

$$d\nu = d\lambda + f\,d\mu$$

be the Lebesgue decomposition.

Then, for μ-a.e. x,

$$f(x) = \lim_{r \to 0} \frac{\nu(E_r)}{\mu(E_r)}$$

for any family $\{E_r\}_{r>0}$ that (shrinks) nicely to x.

Pf: Let $X = E \cup F$ be a partition into μ-null and λ-null sets.

Define

$$F_K := \left\{ x \in F : \limsup_{r \to 0} \frac{|\lambda|(\overline{B_r(x)})}{\mu(\overline{B_r(x)})} > \frac{1}{K} \right\}$$

Claim: $\mu(F_K) = 0 \ \forall K$.

It remains to show the claim. Since $0 = |\lambda|(F) \geq |\lambda|(F_K)$ and $|\lambda|$ is outer regular, $\exists\, \varepsilon > 0$, $\exists\, \mathcal{U}_\varepsilon \supseteq F_K$, $\mathcal{U}_\varepsilon$ open, s.t. $|\lambda|(\mathcal{U}_\varepsilon) < \varepsilon$.

For each $x \in F_K$, $\exists\, r_x > 0$ s.t. $B_{r_x}(x) \subseteq \mathcal{U}_\varepsilon$. Furthermore, $\exists\, \tilde{r}_x \leq r_x$ s.t.

$$|\lambda|\left(\overline{B_{\tilde{r}_x}(x)}\right) > \frac{1}{k}\,\mu\left(\overline{B_{\tilde{r}_x}(x)}\right).$$

Let $\tilde{\mathcal{F}} = \{ \overline{B_{\tilde{r}_x}(x)} : x \in F_K \}$.

By Besicovitch,

$$\begin{aligned} \mu(F_K) &\leq \sum_{j=1}^{\xi(n)} \sum_{\overline{B} \in \tilde{\mathcal{F}}_j} \mu(\overline{B}) \\ &\leq \sum_{j=1}^{\xi(n)} \sum_{\overline{B} \in \tilde{\mathcal{F}}_j} k|\lambda|(\overline{B}) \\ &\leq k \sum_{j=1}^{\xi(n)} |\lambda|(\mathcal{U}_\varepsilon) \leq k\,\xi(n)\,\varepsilon. \end{aligned}$$

Therefore, $\mu(F_K)=0$.

First application of Lebesgue-Besicovitch differentiation: use

"differentiability of measures" $\Rightarrow$ weak diff of functions

one onedimensional Euclidean space

What is the correspondence between measures and functions?

Recall: Given $F:\mathbb{R}\to\mathbb{R}$ increasing, right cts, and $F(-\infty)=0$, $\exists!$ locally finite Borel measure μ_F satisfying

$$\mu_F((-\infty,x]) = F(x), \quad \forall x \in \mathbb{R}.$$

Furthermore, any finite Borel measure is of this form for some F.

Abuse of notation: Given $F:\mathbb{R}\to\overline{\mathbb{R}}$

$$F(-\infty) := \lim_{x\to-\infty} F(x)$$

$$F(+\infty) := \lim_{x\to+\infty} F(x),$$

as long as these limits exist.

Q: What is the analogue of this correspondence for signed measures?

A: Functions of "bounded variation".

Def: Given $F:\mathbb{R}\to\mathbb{R}$, its **total variation** is the function $T_F:\mathbb{R}\to[0,+\infty]$ defined by

$$T_F(x) = \sup\left\{\sum_{j=1}^{n}|F(x_j)-F(x_{j-1})| : n\in\mathbb{N},\ -\infty<x_0<\ldots<x_n=x\right\}$$

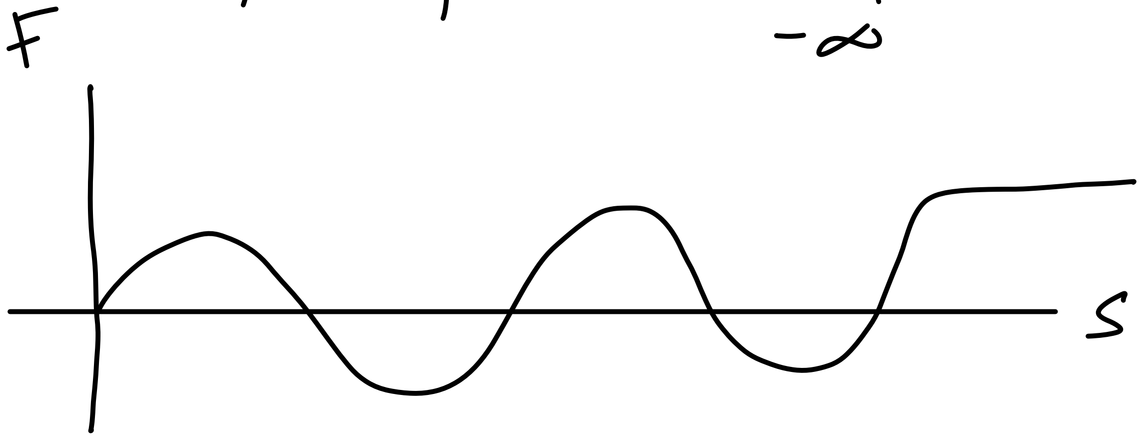
Rmk: If $F:\mathbb{R}\to\mathbb{R}$ is smooth and all derivatives bdd,

$$T_F(x) = \sup\left\{\overbrace{\sum_{j=1}^{n}|F(x_j)-F(x_{j-1})|}^{\sum_j \left|\int_{x_{j-1}}^{x_j} F'(s)\,ds\right| \le \int_{-\infty}^{x}|F'(s)|\,ds} : n\in\mathbb{N},\ -\infty<x_0<\ldots<x_n=x\right\}$$

OTOH, we may express (*) as

$$\overbrace{\sum_j (x_j - x_{j-1}) \underbrace{\left| \int_{x_{j-1}}^{x_j} F'(s)\,ds \right|}_{\xrightarrow[\text{partitions refined}]{} |F'(s)|}}^{\text{Riemann sum approx for } \int_{-\infty}^{x} |F'(s)|\,ds}$$

Thus, $T_F(x) = \int_{-\infty}^{x} |F'(s)|\,ds$



$T_F(x)$ represents the total distance traveled by particle up to tim x.

Rmk:

(i) adding a point x_i to the partition increases value of objetive fn

(ii) If $a<b$, we may assume that a is a point in every partition used to compute $T_F(b)$. Thus $T_F(b) \geq T_F(a)$, so it is increasing.

Def: $F: \mathbb{R} \to \mathbb{R}$ is of **bounded variation** if $T_F(+\infty) < +\infty$. Let BV denote the set of all such functions.

Ex: $\forall x_0 \leq x$, $|F(x_0) - F(x)| \leq T_F(x) \leq T_F(+\infty)$

Thus $F \in BV \Rightarrow F$ is bounded.

Ex: $F(x) = \sin(x)$ is bounded but (heuristically) $F \notin BV$

Ex: If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, $T_F(x) = F(x) - F(-\infty)$ and $F \in BV$.

Ex: If $F, G \in BV$, $a, b \in \mathbb{R}$, then $aF + bG \in BV$.

Thm: $F \in BV$ iff $F = F_1 - F_2$ for F_i bdd, increasing.

We may take $F_1 = \frac{1}{2}(T_F + F)$

$F_2 = \frac{1}{2}(T_F - F)$

Lemma: If $f \in BV$, then $T_F(-\infty)=0$

Pf: For all $x \in \mathbb{R}$, $\varepsilon > 0$, $\exists$ $x_0 < \ldots < x_n = x$ s.t.

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(x) - \varepsilon$$

Furthermore,

$$T_F(x) \geq \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + T_F(x_0)$$

Thus, $\cancel{T_F(x)} - T_F(x_0) \geq \cancel{T_F(x)} - \varepsilon$, so $\varepsilon > T_F(x_0)$. $\square$

Now, we can prove the theorem.

Pf: "$\Leftarrow$" follows by previous two examples.

We show "$\Rightarrow$". Since

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F),$$

it suffices to show $T_F \pm F$ are bdd and increasing.

We will show $T_F + F$ is bdd and increasing. Then we have $T_{-F} + (-F) = T_F - F$ is also.

Since

$$0 = T_F(-\infty) \leq T_F(x) \leq T_F(+\infty) < +\infty,$$

T_F is bounded, so $T_F + F$ is bdd.

Now, we show $x < y \Rightarrow$

$$T_F(x) + F(x) \leq T_F(y) + F(y).$$

Fix $\varepsilon > 0$ arbitrary. Then $\exists$ $x_0 < \dots < x_n = x$ s.t.

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(x) - \varepsilon$$

Thus, $+F(y)$ $\quad +F(y)$

$$T_F(y) \geq \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)|$$

Note: $\overbrace{|F(y) - F(x)| + F(y)} \geq F(x)$

$\boxed{F(x) \geq F(y)}$ $= F(x) - F(y) + F(y) = F(x)$

$\boxed{F(y) \geq F(x)}$ $= F(y) - F(x) + F(y) \geq F(y) \geq F(x)$

This gives

$$T_F(y) + F(y) \geq T_F(x) - \varepsilon + F(x)$$

Since $\varepsilon > 0$ arb, this gives result. $\square$

Now, we can prove main result...

Thm: Given $F \in BV$, right cts, $F(-\infty)=0$, $\exists!$ finite signed Borel measure μ_F s.t.

$$F(x) = \mu_F((-\infty, x]), \ \forall x \in \mathbb{R}.$$

Furthermore, any finite signed Borel measure ν on $\mathbb{R}$ is of this form for $F(x) = \nu((-\infty, x])$.

Lastly, $|\mu_F| = \mu_{T_F}$.

One last lemma.-

Lemma: If $F: \mathbb{R} \to \mathbb{R}$ right cts, so is T_F.

Pf: Exercise.

Pf of Thm: By previous thm,

$$F = \underbrace{\frac{1}{2}(T_F + F)}_{F_1} - \underbrace{\frac{1}{2}(T_F - F)}_{F_2}$$

where F_i bdd, increasing, right cts, $F_i(-\infty) = 0$. Thus $\exists$ finite Borel measures $\mu_i((-\infty, x]) = F_i(x)$, $i = 1, 2$.

Define $\mu_F := \mu_1 - \mu_2$. Then $\mu_F((-\infty, x]) = F(x)$.

Furthermore, signed measures are uniquely determined by their values on such intervals.

Conversely, given ν as in the theorem,

$$\nu = \nu^+ - \nu^-,$$

where $\nu^\pm$ finite positive measures. Thus $F^\pm(x) = \nu^\pm((-\infty, -x])$ are right cts, increasing, $F^\pm(-\infty)$, so $F := F^+ - F^-$ satisfies required properties. $\square$