Lecture 9

260R, Advanced Measure Theory
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Reminder: Presentation Topics - May 5
- May 14

Def: Given $F: \mathbb{R} \to \mathbb{R}$, its **total variation** is the function $T_F : \mathbb{R} \to [0, +\infty]$ defined by

$$T_F(x) = \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \dots < x_n = x \right\}$$

Rmk: If $F:\mathbb{R}\to\mathbb{R}$ is smooth and all derivatives bdd,

$$T_F(x) = \int_{-\infty}^{x} |F'(s)|\,ds$$

Rmk: $a<b$

$$T_F(b) \geq T_F(a) + |F(b)-F(a)|$$

Def: $F:\mathbb{R}\to\mathbb{R}$ is of bounded variation if $T_F(+\infty)<+\infty$. Let BV denote the set of all such functions.

Ex: $F \in BV \Rightarrow F$ is bounded.

Ex: If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, $T_F(x) = F(x) - F(-\infty)$ and $F \in BV$.

Ex: If $F, G \in BV$, $a, b \in \mathbb{R}$, then $aF + bG \in BV$.

Thm: $F \in BV$ iff $F = F_1 - F_2$ for F_i bdd, increasing.
We may take
$$F_1 = \frac{1}{2}(T_F + F)$$
$$F_2 = \frac{1}{2}(T_F - F)$$

Lemma: If $F \in BV$, then $T_F(-\infty) = 0$

Lemma: If $F : \mathbb{R} \to \mathbb{R}$ right cts, so is T_F.

Thm: Given $F \in BV$, right cts, $F(-\infty) = 0$, $\exists!$ finite signed Borel measure μ_F s.t.

$$F(x) = \mu_F((-\infty, x]), \ \forall x \in \mathbb{R}.$$

Furthermore, any finite signed Borel measure ν on $\mathbb{R}$ is of this form for $F(x) := \nu((-\infty, x])$.

Lastly, $|\mu_F| = \mu_{T_F}$.

Recall: Partition Formula

$\forall \ E \in \mathcal{B}_{\mathbb{R}}$

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{n} |\mu(E_i)| : E = \bigsqcup_{i=1}^{n} E_i, \ E_i \in \mathcal{B}_{\mathbb{R}}, \ n \in \mathbb{N} \right\}$$

Note that this is not $|\mu|(E_i)$

Pf: We now show $|\mu_F| = \mu_{T_F}$.

Step 1: $|\mu_F| \geq \mu_{T_F}$.

Since finite Borel measures are uniquely determined by their CDF, it suffices to show

$$G(x) := |\mu_F|((-\infty, x]) \geq \mu_{T_F}((-\infty, x]) = T_F(x) \quad \forall x \in \mathbb{R}$$

Since,

$$|\mu_F|(x_{i-1}, x_i] \geq |\mu_F((x_{i-1}, x_i])| = |F(x_i) - F(x_{i-1})|$$

we may estimate...

$T_F(x) =$

$$\sup\left\{\sum_{j=1}^{n}|F(x_j)-F(x_{j-1})| : n\in\mathbb{N}, -\infty<x_0<\dots<x_n=x\right.$$

$$\le \sup\left\{\sum_{j=1}^{n}|\mu_F|((x_{j-1},x_j]) : n\in\mathbb{N}, -\infty<x_0<\dots<x_n=x\right.$$

$$\swarrow \text{ countable additivity}$$

$$= |\mu|_F(-\infty, x]$$

$$= G(x)$$

Step 2: $\mu_{T_F} \geq |\mu_F|$

$$|\mu_F((-\infty,b]) - \mu_F((-\infty,a])|$$

$$\pm\mu_F((a,b])$$

$$\rotatebox{-90}{$\le$}$$

$$|\mu_F((a,b])| = |F(b)-F(a)|$$

$$\le T_F(b) - T_F(a)$$

$$= \mu_{T_F}((a,b])$$

Thus $(\mu_{T_F} \pm \mu_F)((a,b]) \geq 0 \ \forall a<b$.

Since L-S measures are characterized by values on half open intervals,

$$(\mu_{T_F} \pm \mu_F)(E) \geq 0, \ \forall E \in \mathcal{B}_{\mathbb{R}}.$$

$$\Updownarrow$$

$$\mu_{T_F}(E) \geq \pm \mu_F(E)$$

$$\Updownarrow$$

$$\mu_{T_F}(E) \geq |\mu_F(E)|$$

By Partition Formula,

$$|\mu|(E) = \sup\left\{ \sum_{i=1}^{n} |\mu(E_i)| : E = \bigsqcup_{i=1}^{n} E_i, \ E_i \in \mathcal{B}_{\mathbb{R}}, \ n \in \mathbb{N} \right\}$$

$$\leq \sup \left\{ \sum_{i=1}^{n} \mu_{T_F}(E_i) : E = \bigsqcup_{i=1}^{n} E_i,\ E_i \in \mathcal{B}_{\mathbb{R}},\ n \in \mathbb{N} \right\}$$

$$= \mu_{T_F}(E)$$

□

Given F, when is $\mu_F \ll \mathcal{L}$?

Prop: Given $F \in BV$, right cts, $F(-\infty) = 0$, then F' exists $\mathcal{L}$-a.e. and $F' \in L^1(\mathcal{L})$.

Furthermore,

$\mu_F \perp \mathcal{L} \iff F' = 0$ $\mathcal{L}$-a.e.

$\mu_F \ll \mathcal{L} \iff F(x) = \int_{-\infty}^{x} F'(t)\, d\mathcal{L}(t)$.

Pf: Let μ_F be corresponding signed measure.

$$\frac{F(x+h)-F(x)}{h} = \begin{cases} \frac{1}{h}\mu_F((x,x+h]), & h>0 \\ \frac{-1}{h}\mu_F((x+h,x]), & h<0 \end{cases}$$

$$= \begin{cases} \frac{\mu_F((x,x+h])}{\mathcal{L}((x,x+h])} & h>0 \\ \text{''} \qquad \text{''} & \text{''} \end{cases}$$

since $(x,x+h], (x+h,x]$ shrink nicely to x, as $h \to 0$ this converges for $\mathcal{L}$-a.e. x to $f \in L^1(\mathcal{L})$ where

$$d\mu_F = d\lambda + f\,d\mathcal{L}.$$

Thus F' exists $\mathcal{L}$-a.e. and $F' \in L^1(\mathcal{L})$.

If $\mu_F \perp \mathcal{L}$, we showed before $f=0$.

If $\mu_F << \mathcal{L}$, we showed $d\mu_F = f\,d\mathcal{L}$, and this implies

$$F(x) = \mu_F((-\infty, x]) = \int_{-\infty}^{x} f\,d\mathcal{L}$$

$$= \int_{-\infty}^{x} F'(t)\,d\mathcal{L}(t)$$

$\square$

Just like there is a correspondence between total variation of measure and total variation of function, also correspondence between notions of absolute continuity.

Def: $F: \mathbb{R} \to \mathbb{R}$ is absolutely continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for all finite collections of disjoint intervals $(a_1, b_1), \dots, (a_N, b_N)$,

$$\sum_{i=1}^{N} |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^{N} |F(b_j) - F(a_i)| < \varepsilon.$$

Ex: F is abs cts $\Rightarrow$ F unif cts
$(N=1)$

Prop: Given $F \in BV$, right cts, $F(-\infty)$,
F is abs cts $\Leftrightarrow \mu_F \ll \mathcal{L}$.

Pf: First, suppose $\mu_F \ll \mathcal{L}$.
We showed that $|\mu_F| \ll \mathcal{L}$.

For all $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.
$\mathcal{L}(E) < \delta$ for some $E \in \mathcal{B}_{\mathbb{R}}$
$\Longrightarrow |\mu_F|(E) < \varepsilon$.

Taking $E = \bigsqcup_{i=1}^{N} (a_i, b_i)$, we see that
$\mathcal{L}(E) < \delta \Longleftrightarrow \sum_{i=1}^{N} |b_i - a_i| < \delta$ and

$$\varepsilon > |\mu_F|(E) = \sum_{i=1}^{N} |\mu_F|(a_i, b_i)$$
$$\geq \sum_{i=1}^{N} |\mu_F(a_i, b_i)|$$
$$= \sum_{i=1}^{N} |F(b_i) - F(a_i)|,$$

Will show other direction next time ☺