Through out this notes μ is a Radon measure on $\mathbb{R}^n$.

1. Concentrated Measures

Definition. We say that μ is *concentrated on* a measurable set A if

$$\mu(\mathbb{R}^n \setminus A) = 0.$$

Equivalently, for every measurable set E,

$$\mu(E) = \mu(E \cap A).$$

Example. The Dirac measure δ_{x_0} is concentrated on $\{x_0\}$, since

$$\delta_{x_0}(\mathbb{R}^n \setminus \{x_0\}) = 0.$$

2. Support of a Measure

Definition. The support of μ, denoted $\operatorname{spt}(\mu)$, is the smallest closed set on which μ is concentrated.

Equivalently,

$$\operatorname{spt}(\mu) = \{x \in \mathbb{R}^n : \mu(B_r(x)) > 0 \text{ for every } r > 0\}.$$

Proof of equivalence. Let

$$S = \{x \in \mathbb{R}^n : \mu(B_r(x)) > 0 \text{ for every } r > 0\}.$$

If $x \notin S$, then some ball $B_r(x)$ has measure zero. Every point near x also has a smaller ball contained in $B_r(x)$, so the complement of S is open. Hence S is closed.

Also, $\mathbb{R}^n \setminus S$ is a union of open balls of measure zero. Since $\mathbb{R}^n$ is second-countable, this union may be reduced to a countable union, so

$$\mu(\mathbb{R}^n \setminus S) = 0.$$

Thus μ is concentrated on the closed set S.

Now let C be any closed set such that $\mu(\mathbb{R}^n \setminus C) = 0$. If $x \notin C$, then since C is closed, some ball $B_r(x)$ lies inside $\mathbb{R}^n \setminus C$. Therefore

$$\mu(B_r(x)) = 0,$$

so $x \notin S$. Hence $S \subset C$. Therefore S is the smallest closed set on which μ is concentrated, so

$$S = \operatorname{spt}(\mu).$$

Example. Let $\{q_n\}_{n=1}^{\infty}$ enumerate $\mathbb{Q}$ and define

$$\mu = \sum_{n=1}^{\infty} \delta_{q_n}.$$

Then μ is concentrated on $\mathbb{Q}$, but

$$\operatorname{spt}(\mu) = \mathbb{R},$$

because every nonempty open interval contains some rational q_n, hence has positive μ-measure.

3. Relation to Lebesgue Decomposition

By the Lebesgue decomposition theorem,

$$\mu = \mu_{ac} + \mu_s,$$

where

$$\mu_{ac} \ll \mathcal{L}^n, \qquad \mu_s \perp \mathcal{L}^n.$$

The singularity condition $\mu_s \perp \mathcal{L}^n$ means that there exists a Borel set N such that

$$\mathcal{L}^n(N) = 0, \qquad \mu_s(\mathbb{R}^n \setminus N) = 0.$$

Thus μ_s is concentrated on a Lebesgue-null set.
However, concentration is measure-theoretic, while support is topological. A singular measure can be concentrated on a null set but still have full support, as in the example above.

4. Discrete and Continuous Parts

The singular part can be further decomposed as

$$\mu_s = \mu_{pp} + \mu_{sc},$$

where μ_{pp} is the pure point part and μ_{sc} is the singular continuous part.
Thus

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}.$$

Here

$$\mu_{pp} = \sum_{x:\, \mu(\{x\})>0} \mu(\{x\})\delta_x,$$

so μ_{pp} is concentrated on a countable set of atoms.

The measure μ_{sc} is singular with respect to Lebesgue measure and has no atoms:

$$\mu_{sc} \perp \mathcal{L}^n, \qquad \mu_{sc}(\{x\}) = 0 \quad \text{for every } x.$$