These notes are adapted from a presentation on the Vitali Property on May 5, 2026 for a course on Advanced Measure Theory with Prof. Katy Craig at UCSB.

Recall the following Theorem about covering a subset of $\mathbb{R}^n$ using closed nondegenerate balls:

Theorem (Besicovitch Covering Theorem)**.** *Let $\mathcal{F}$ be a collection of closed nondegenerate balls in $\mathbb{R}^n$, and let C be the set of centers of the balls in $\mathcal{F}$. Suppose either that*

1. *C is bounded, or*
2. $\sup\{\mathrm{diam}(B) : B \in \mathcal{F}\} < \infty$

There is a constant $\xi(n)$, depending only on n, so that C can be covered by at most $\xi(n)$ countable disjoint subcollections $\mathcal{F}_i \subset \mathcal{F}$.

Note that the Theorem makes no reference to any type of measure that you could put on $\mathbb{R}^n$. The goal of these notes is to prove that if we give $\mathbb{R}^n$ a locally finite Borel regular measure μ, and impose some extra conditions on $\mathcal{F}$ and on C, there is a single countable disjoint subcollection that covers almost all of C:

Corollary (Vitali Property)**.** Let $\mathcal{F}, C$ be as in the Besicovitch Covering Theorem, and let μ be a locally finite Borel regular measure on $\mathbb{R}^n$. Suppose that C is both bounded and μ-measurable, and that the following condition holds for all $x \in C$:

$$\inf\{\mathrm{diam}(B) : B \in \mathcal{F}, x \text{ is the center of } B\} = 0. \tag{$*$}$$

Then there exists a countable disjoint subcollection $\mathcal{G} \subset \mathcal{F}$ so that

$$\mu\left(C \setminus \left(\bigcup_{B \in \mathcal{G}} B\right)\right) = 0.$$

Before we prove the Corollary we prove one Lemma (which is really a Corollary to the Besicovitch Covering Theorem)

Lemma. *Let μ be an outer measure on $\mathbb{R}^n$, and let $\mathcal{F}, C$ be as in the Besicovitch Covering Theorem. There exists a countable disjoint subcollection $\mathcal{F}' \subset \mathcal{F}$ so that*

$$\mu(C) \leq \xi(n) \sum_{B \in \mathcal{F}'} \mu(C \cap B).$$

Additionally if μ is a Borel measure and C is μ-measurable, then

$$\mu(C) \leq \xi(n)\mu\left(C \cap \left(\bigcup_{B \in \mathcal{F}'}\right)\right).$$

Proof. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{\xi(n)}$ be the countable disjoint subcollections of $\mathcal{F}$ given by the Besicovitch Covering Theorem, and let $\mathcal{G} = \bigcup_{i=1}^{\xi(n)} \mathcal{F}_i$. Then since $\mathcal{G}$ covers C, we have $\bigcup_{B \in \mathcal{G}} C \cap B = C$, so by countable subadditivity of μ we have

$$\mu(C) \leq \sum_{B \in \mathcal{G}} \mu(C \cap B) = \sum_{i=1}^{\xi(n)} \sum_{B \in \mathcal{F}_i} \mu(C \cap B).$$

Pick the index k that maximizes the interior sum, and set $\mathcal{F}' = \mathcal{F}_k$. Then we have

$$\mu(C) \leq \xi(n) \sum_{B \in \mathcal{F}'} \mu(C \cap B).$$

Now if μ is a Borel measure and C is μ-measurable, then $C \cap \left(\bigcup_{B \in \mathcal{F}'} B\right) = \bigcup_{B \in \mathcal{F}'} C \cap B$ is a countable disjoint union of μ-measurable sets. Then by countable additivity we have

$$\mu(C) \leq \xi(n) \mu \left(C \cap \left(\bigcup_{B \in \mathcal{F}'} B \right) \right).$$

□

Proof of Vitali Property: Say that the pair $(\mathcal{F}, C)$ of a collection of nondegenerate closed balls $\mathcal{F}$ and the set of centers C is a *Vitali pair* if they satisfy all of the requirements for the Vitali Property. In particular C is bounded and μ-measurable and every element of C satisfies $(*)$. By the Lemma there is a countable disjoint subcollection $\mathcal{F}' \subset \mathcal{F}$ so that

$$\frac{\mu(C)}{\xi(n)} \leq \mu \left(C \cap \left(\bigcup_{B \in \mathcal{F}'} B \right) \right).$$

We can write $\mathcal{F}' = \{B_k\}_{k \in \mathbb{N}}$. The sets $C \cap (\bigcup_{k=1}^n B_k) = \bigcup_{k=1}^n C \cap B_k$ form an increasing sequence in n of sets that converges to $C \cap (\bigcup_{k=1}^\infty B_k)$. Since C is bounded and μ is locally finite, we have $\mu(C) < \infty$. Using continuity from below we have that $\mu\left(C \cap (\bigcup_{k=1}^n B_k)\right) \nearrow \mu\left(C \cap (\bigcup_{k=1}^\infty B_k)\right)$. Then for some $N_1 \in \mathbb{N}$ we have that

$$\frac{\mu(C)}{2\xi(n)} \leq \mu \left(C \cap \left(\bigcup_{k=1}^{N_1} B_k \right) \right).$$

Since $\mu(C) = \mu\left(C \cap \left(\bigcup_{k=1}^{N_1} B_i\right)\right) + \mu\left(C \setminus \left(\bigcup_{k=1}^{N_1} B_k\right)\right)$, letting $\theta = 1 - (2\xi(n))^{-1}$ we have that $\mu\left(C \setminus \left(\bigcup_{k=1}^{N_1} B_k\right)\right) \leq \theta \mu(C)$.

Now let $\mathcal{F}_1 = \{B \in \mathcal{F} : B \cap B_k = \emptyset, k = 1, \ldots, N_1\}$, and let $C_1 = C \setminus \left(\bigcup_{k=1}^{N_1} B_k\right)$. If $B \in \mathcal{F}_1$, then $B \cap B_k = \emptyset$ for all $k = 1, \ldots, N_1$ implies that the center of B is not in any of the B_k, so the center of B is in C_1. On the other hand, if $x \in C_1$, then since there are only finitely many B_k and condition $(*)$ holds, there is some ball in $\mathcal{F}_1$ whose center is x (namely, pick a ball with center x whose radius is less than half the distance from x to the union of the B_k). Thus we have shown that C_1 is the set of centers of the balls in $\mathcal{F}_1$. Since $C_1 \subset C$ is bounded and μ-measurable, and condition $(*)$ still holds for $\mathcal{F}_1$, we have that $(\mathcal{F}_1, C_1)$ is also a Vitali pair. Hence we can increase our initial collection of N_1 disjoint balls to a collection of $N_2 > N_1$ disjoint balls so that $\mu\left(C \cap \left(\bigcup_{k=1}^{N_2} B_k\right)\right) \leq \theta \mu(C_1) \leq \theta^2 \mu(C)$.

By repeatedly generating Vitali pairs we get a sequence of positive integers $N_h \to \infty$, increasing finite collections $\{B_k\}_{k=1}^{N_h}$ of disjoint balls in $\mathcal{F}$, and μ-measurable sets $C_h = C \setminus \left(\bigcup_{k=1}^{N_h} B_k\right)$ with $\mu(C_h) \leq \theta^h \mu(C)$. Since the C_h are also a decreasing sequence of μ-measurable sets all contained in C, with $\mu(C) < \infty$, continuity from above implies that

$$\mu\left(C \setminus \left(\bigcup_{k=1}^{\infty} B_k\right)\right) = \lim_{h\to\infty} \mu(C_h) \leq \lim_{h\to\infty} \theta^h \mu(C) = 0.$$

It follows that the collection $\mathcal{G} = \{B_k\}_{k\in\mathbb{N}}$ is the countable disjoint collection we want. $\square$