Let $\mu$ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha\in A} \subseteq \mathcal{L}^1(\mu)$ is called uniformly integrable if

$\forall \epsilon>0$, $\exists \delta>0$ s.t. $\mu(E)<\delta \Rightarrow \sup_\alpha \left|\int_E f_\alpha \, d\mu\right| < \epsilon$

We have some following exercise below:

(a). Any finite subset of $\mathcal{L}^1(\mu)$ is U.I.

Pf: (Base case) It is straight forward to see $\{f\}$ is u.i for $f\in\mathcal{L}^1(\mu)$.

(Inductive Step): Suppose $\{f_k\}_{k=1}^n \subseteq \mathcal{L}^1(\mu)$ for some $n\in\mathbb{N}$ satisfies $\{f_k\}_{k=1}^n$ is u.i. Then, for every $\varepsilon>0$, choose $\delta_1>0$ s.t. $\mu(E)<\delta_1 \Rightarrow \sup_{1\le k\le n} \left|\int_E f_k \, d\mu\right| < \epsilon$

Consider $f_{n+1}\in\mathcal{L}^1(\mu)$ so that we can choose $\delta_2>0$ s.t. $\mu(E)<\delta_2 \Rightarrow \left|\int_E f_{n+1}\, d\mu\right| < \epsilon$

Take $\delta = \min\{\delta_1,\delta_2\}$, we have

$$\mu(E)<\delta \Rightarrow \sup_{1\le k\le n+1} \left|\int_E f_k \, d\mu\right|$$

$$= \sup\left\{ \sup_{1\le k\le n} \left|\int_E f_k\, d\mu\right|, \left|\int_E f_{n+1}\, d\mu\right| \right\} < \epsilon$$

which completes the induction proof. $\blacksquare$

(b) If $\{f_n\} \subset \mathcal{L}^1(\mu)$ s.t. $f_n \xrightarrow{\mathcal{L}^1} f$ $\left(\int |f-f_n| d\mu \xrightarrow{n\to\infty} 0\right)$, $f \in \mathcal{L}^1(\mu)$

then $\{f_n\}$ is u.I.

Pf: $f_n \xrightarrow{\mathcal{L}^1} f \in \mathcal{L}^1(\mu)$ implies for every $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ s.t.

$n > N \Rightarrow \int |f - f_n| d\mu < \epsilon/2$

By (a), we showed that $\{f_n\}_{n=1}^N$ is u.i. since $f_n \in \mathcal{L}^1$ $\forall n$.

Thus, we can choose $\delta_1 > 0$ s.t. $\mu(E) < \delta_1 \Rightarrow \sup_{1 \le n \le N} \left| \int_E f_n d\mu \right| < \epsilon$.

Also, for $n > N$, write

$$\int_E |f_n| \, d\mu \le \int_E |f_n - f| d\mu + \int_E |f| d\mu < \frac{\epsilon}{2} + \int_E |f| d\mu$$

$f \in \mathcal{L}^1$ implies that we can choose $\delta_2$ s.t. $\mu(E) < \delta_2 \Rightarrow \int_E |f| d\mu < \frac{\epsilon}{2}$

Thus, take $\delta = \min\{\delta_1, \delta_2\}$, $\mu(E) < \delta \Rightarrow \sup_n \left| \int_E f_n d\mu \right| < \epsilon$, completed $\boxed{S}$

Rmk:

(1) The converse is not true. Consider $X = [0,1]$ equiped with Lebesgue measure, $f_k = \begin{cases} 1_{[0,1]} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$

$\{f_k\}$ is u.i. but not $\mathcal{L}^1$ convergence.

(2) The following theorem weakens the statements that $f_n \xrightarrow{\mathcal{L}^1} f$ under some assumptions, so that the logically equivalent statements hold.

# Standard Dunford Pettis Theorem

$(X, \mathcal{M}, \mu)$ — a finite measure $\mu$, a bounded seq $\{f_n\} \subseteq \mathcal{L}^1(\mu)$,

the following two properties are equivalent:

(i) $\{f_n\}$ is u.i.

(ii) Every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $\mathcal{L}^1(\mu)$. $\Rightarrow$ (Weakly Compactness)

Weak Convergence: $f_n \rightharpoonup f$ weakly in $\mathcal{L}^1$ if for every $g \in \mathcal{L}^\infty$, $\int g f_n \, d\mu \to \int g f \, d\mu$.

Rmk: $\mathcal{L}^\infty$ is known as $(\mathcal{L}^1)^*$, which is the dual space of $\mathcal{L}^1$ in functional analysis classes.

Vitali Criterion: $\{f_\alpha\}_{\alpha \in A}$ is $\mu$-u.i iff

finite

(1) $\sup_\alpha \int_X |f_\alpha| \, d\mu < \infty$

(2) $\forall \varepsilon > 0$, $\exists M > 0$ s.t. $\sup_\alpha \int_{\{|f_\alpha| > M\}} |f_\alpha| \, d\mu < \epsilon$

Pf: ($\Rightarrow$) $\forall \varepsilon > 0$, we can choose $\delta > 0$ s.t.

$\mu(E) < \delta \Rightarrow \sup_\alpha \int_E |f_\alpha| \, d\mu < \epsilon$.

Let $X = E_1 \cup E_2 \cup \cdots \cup E_m$ in which $\mu(E_i) \leq \frac{\delta}{2}$ $\forall i$

and $m \in \mathbb{N}$ (possible since $\mu(X) < \infty$)

Then, for all $\alpha$:

$$\int_X |f_\alpha|\,d\mu \le \sum_{i=1}^{m} \int_{E_i} |f_\alpha|\,d\mu < \overset{K}{\overset{\shortparallel}{m\varepsilon}} \Rightarrow \sup_\alpha \int_X |f_\alpha|\,d\mu < \infty.$$

Since $\mu$ is a finite measure and $\sup_\alpha \int_X |f_\alpha|\,d\mu < \infty$,

$$\mu(\{|f_\alpha| > M\}) \le \frac{\int_X |f_\alpha|\,d\mu}{M} \le \frac{K}{M} \quad \text{(Markov's ineq)}$$

Choose $M$ s.t. $\frac{K}{M} < \delta$, so $\mu(\{|f_\alpha| > M\}) < \delta$,

which implies $\int_{\{|f_\alpha| > M\}} |f_\alpha|\,d\mu < \epsilon$,

it holds for all $\alpha$, so $\sup_\alpha \int_{\{|f_\alpha| < M\}} |f_\alpha|\,d\mu < \epsilon$.

($\Leftarrow$) for any measurable set $A$:

$$\int_A |f_\alpha|\,d\mu = \int_{A \cap \{|f_\alpha| > M\}} |f_\alpha|\,d\mu + \int_{A \cap \{|f_\alpha| \le M\}} |f_\alpha|\,d\mu$$

$$\le M\mu(A) + \int_{\{|f_\alpha| > M\}} |f_\alpha|\,d\mu$$

For any $\varepsilon > 0$, choose $M$ s.t. $\sup_\alpha \int_{\{|f_\alpha| > M\}} |f_\alpha|\,d\mu < \epsilon/2$

Choose $\delta = \frac{\varepsilon}{2M}$. Then, $\mu(A) < \delta$ implies $M\mu(A) < \infty$

$$\Rightarrow \sup_\alpha \int_A |f_\alpha|\,d\mu \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon.$$