Motivation of LOT

- Given $\{(X_i, y_i)\}_{i=1}^{n}$

$$X_i = \{x_k^{(i)}\}_{k=1}^{n} \subseteq \mathbb{R}^d$$

$$y_i \in \{-1, 1\}$$

- Cell Bio
- shape statistics

LOT Alg

Fix $\sigma \in P_2(\mathbb{R}^d)$

$$T_\sigma^{\mu_i} = \arg\min_T \int \|T(x) - x\|^2 \, d\sigma(x)$$

$$\mu_i \mapsto T_\sigma^{\mu_i}$$

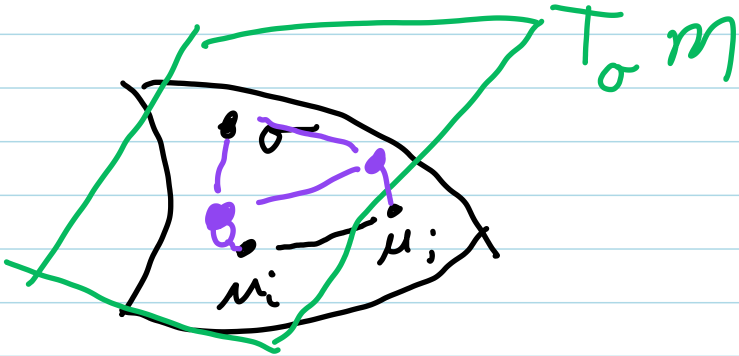
$$\wr$$

$$X_i \mapsto T_\sigma^{X_i} \in \mathbb{R}^{m \times d}$$

$\uparrow$ m samples

SVM: $\quad y_i \langle T_\sigma^{X_i}, w \rangle \geq 1$

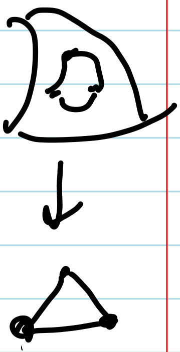
Vasserstein Manifold



Curvature $\quad (S, d)$

Triangle $\{p, x, y\} \subseteq S$ $\qquad$ isometric embed

Def: $\operatorname{curv}(S) \geq 0$ if $\forall$ const speed geodesics



$\omega(t)$ from x to y, $\bar{\omega}(t)$ on $\{\bar{p}, \bar{x}, \bar{y}\}$

$$d(p, \omega(t)) \geq \|\bar{p} - \bar{\omega}(t)\|$$

$$\Rightarrow d^2(p, \omega(t)) \geq (1-t)\, d^2(p, x) + t\, d^2(p, y) - t(1-t)\, d^2(x, y)$$

$$\Rightarrow d^2(\underset{p \to x}{\omega(s)}, \underset{p \to y}{\omega'(t)}) \geq \|\bar{\omega}(s) - \bar{\omega}'(t)\|^2$$

Wass. curv.

Thm: $\mathrm{curv}(W_2(\mathbb{R})) = 0$

$$W_2^2(\mu,\nu) = \| F_\mu^+(u) - F_\nu^+(u) \|^2$$

Thm: $\mathrm{curv}(W_2(\mathbb{R}^d)) \geq 0$

Pf $W_2^2(\rho, \omega(t)) = \int \|z - v\|^2 \gamma_t(z,v)$

$$= \int \| z - ((1-t)x + ty \|^2 \gamma(dx, dy, dz)$$

$$= (1-t)\|z - x\|^2 + \dots$$

$\geq$

sub opt coupling

Aside $\quad 0<p<1$

$\operatorname{curv}(W_p(\mathbb{R}^d)) \leq 0$

Ramified OT (Xia, UC Davis)

Tree Approx. (Leeb, U Minn)

Angles

$$\cos \measuredangle_p(x, y) = \frac{\|x-p\|^2 + \|y-p\|^2 - \|x-y\|^2}{2\|x-p\| \, \|y-p\|}$$

Generalizes to M by replacing $\| \|$ w/ $d(\cdot,\cdot)$

$$\angle_p(\omega, \omega') = \lim_{s,t \downarrow 0} \angle_p(\omega(s), \omega'(t))$$

On T_pS, $\quad d_p((\omega, s), (\omega', t)) = \sqrt{s^2 + t^2 - 2st\cos\angle(\omega, \omega')}$

Wass angles

$$d_\mu^2\Big(\big(\omega_\nu, d(\mu,\nu)\big), \big(\omega_\rho, d(\mu,\rho)\big)\Big)$$

$$= W_2^2(\mu,\nu) + W_2^2(\mu,\rho)$$

$$- 2W_2(\mu,\nu)\, W_2(\mu,\rho) \cos\measuredangle$$

$$\cos\measuredangle(\omega_\nu, \omega_\rho) = \frac{W_2^2(\mu,\nu) + W_2^2(\mu,\rho) - \dfrac{W_2^2(\omega_{\mu\nu}(t), \omega_\rho(t))}{t^2}}{2\, W_2(\mu,\nu)\, W_2(\mu,\rho)}$$

Lemma: $$\lim_{t \to 0} \frac{W_2^2(\omega_\nu(t), \omega_\rho(t))}{t^2} = \| T_\mu^\nu - T_\mu^\rho \|^2_{L^2(\mu)}$$

$$\cos \angle(\omega_\nu, \omega_\rho) = \cos \measuredangle\left(T_\mu^\nu - Id, T_\mu^\rho - Id\right)$$

$$d_\mu^2 = \| s(T_\mu^\nu - Id) - t(T_\mu^\rho - Id) \|^2_{L^2(\mu)}$$

$T_\mu M$ is isom. to Hilbert space

Thm: $\log_\mu(\nu) = T_\mu^\nu - Id$

Q: What classes can SVM separate?

Q: $\| T_\sigma^\mu - T_\sigma^\nu \| \overset{?}{\approx} W_2(\mu, \nu)$