LOT

$$F_\sigma(\mu) = T_\sigma^{\mu} = \nabla \phi_\mu$$

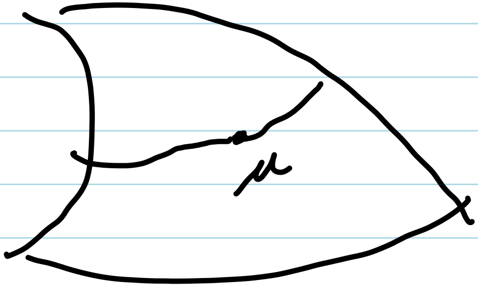
Data gen process

Base dist. μ, ν

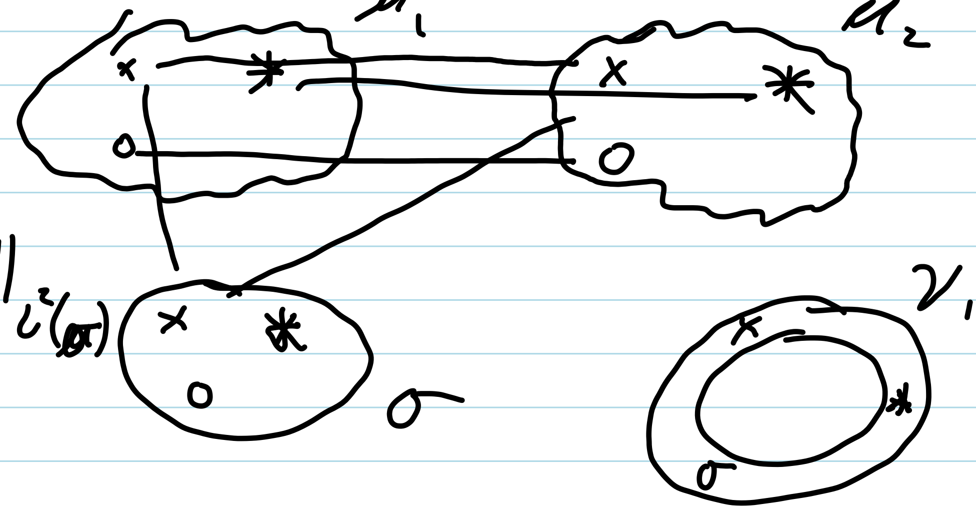
Push forwards: $\mathcal{H}$

Data pt: $h_\# \mu$ for some $h \in \mathcal{H}$

$h_\# \nu$



$$\| T_\sigma^{\mu_1} - T_\sigma^{\mu_2} \|_{L^2(\sigma)}$$



Compatible fctns $\mathcal{H}$

$$
\begin{tikzcd}
\mu \arrow[rr, dash, "h_\#"] \arrow[d, dash, "T_\sigma^\mu"'] & & h_\#\mu \\
\sigma \arrow[urr, dash, "T_\sigma^{h_\#\mu}"'] & &
\end{tikzcd}
$$

$$h \circ T_\sigma^\mu \stackrel{?}{=} T_\sigma^{h_\#\mu}$$

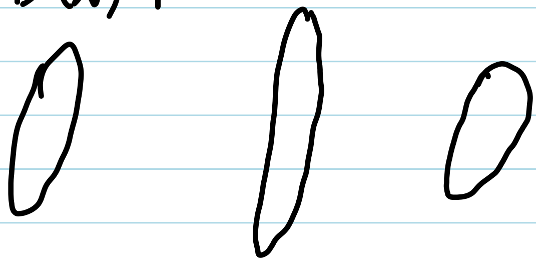
$$h \circ \nabla \phi_\mu \stackrel{?}{=} \nabla \psi$$

$h \in \mathcal{H}$?

- shifts : $h(x) = x + a$
- scalings : $h(x) = c\,x \qquad c > 0$

Asspt on σ, μ

$$J_{T_\sigma \mu}(x) = P^T D(x) P$$



Perturbations

$$\mathcal{H}_{\varepsilon,R} = \{ g \in L^2(\mathbb{R}^d) : \exists\, h \in \mathcal{H} \text{ s.t. } \|g-h\| < \varepsilon \ \& \ \|h\| < R \}$$

Open Q: Interesting deterministic fctns in $\mathcal{H}_{\varepsilon,R}$?

Convexity

Thm. Let $\mathcal{H}$ be convex set of comp. trans. w.r.t. F_σ, μ. Then $F_\sigma(\mathcal{H} * \mu)$ is convex

Thm: Same asspt. Then $F_\sigma(\mathcal{H}_\varepsilon * \mu)$ is 2ε-convex

$$\{F_\sigma(h_\# \mu)\}_{h \in \mathcal{H}_\varepsilon}$$

Classification

Thm: Same asspt. If

- $\mathcal{H} * \mu$, $\mathcal{H} * \nu$ is compact
- $W_2(h_{\#}\mu, g_{\#}\nu) > \delta_\varepsilon \quad \forall h, g \in \mathcal{H}_{\varepsilon,R}$

then $F_\sigma(\mathcal{H}_{\varepsilon,R} * \mu)$ & $F_\sigma(\mathcal{H}_{\varepsilon,R} * \nu)$

are lin. sep.

Thm: Let σ, μ satisfy Caffarelli cond

$g, h \in \mathcal{H}_{\varepsilon, R}$,

$$0 \leq \| T_\sigma^{g\#\mu} - T_\sigma^{h\#\mu} \| - W_2(g_\#\mu, h_\#\nu)$$

$$\leq C \varepsilon^{1/2}$$

$$\delta_\varepsilon = \tilde{C}_{\sigma,\mu} \varepsilon^{1/2}$$

All results can be est. statistically

(Informal) $\|T_\sigma^{\hat{\mu}} - T_\sigma^{\mu}\| \sim \mathcal{O}(n^{-1/d})$

In practice, $\{z_i\}_{i=1}^{n} \sim \sigma$

$\{x_i\}_{i=1}^{n} \sim \mu$

Barycentric proj: Given $\pi \in P(X \times Y)$

$$\hat{T}(x_i) = \frac{1}{\sum_j \pi_{ij}} \sum_j \pi_{ij} y_j$$

$$\hat{T}_\# X \neq Y$$

PCA

Ex: $\{(x+u_i)_\# \mu\}_{i=1}^n$

Thm: Let Y be true low-dim embedding of $\mathcal{H}_\varepsilon$. Then let U be left SV of

$$\left[\hat{T}_\sigma^{h_i} {}_\# \mu\right]_{i=1}^n - \frac{1}{n} 1_n^T [\hat{T}]$$

then w/ prob $1-\eta$

$$\min_{\theta \in O(d)} \| U - Y\theta \| \le C\, n \cdot \|Y^+\| \left(\varepsilon^c + O(n^{-1/d}) \right)$$