Lecture 16

Reminders
- Solutions for 2-3 exercises
- Revise article by March 7th
- Makeup Lecture, Friday, March 14, 9:30-10:45am

Recall:

The Continuity Equation (CE)

$$(CE)\begin{cases} \partial_t \mu_t(x) + \nabla\cdot(\mu_t(x)\, v(x,t)) = 0 \\ \mu_t(x)\big|_{t=0} = \mu_0(x) \end{cases}$$

Def: Given $T>0$, $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$, $v: \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ measurable, $\mu: [0,T] \to \mathcal{M}(\mathbb{R}^d)$ is a weak solution of (CE) if...

(i) $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$,

(a) $t \mapsto \int_{\mathbb{R}^d} \varphi \, d\mu_t$ is abs cts

(b) $\frac{d}{dt}\int_{\mathbb{R}^d} \varphi \, d\mu_t = \int_{\mathbb{R}^d} \nabla\varphi(x)\cdot v(x,t)\, d\mu_t(x)$ for a.e. $t \in [0,T]$

(ii) $\int_0^T \int_{\mathbb{R}^d} |v(x,t)| \, d\mu_t(x) \, dt < +\infty$

Def: (solution of ODE)

Given $v : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ meas, $x_0 \in \mathbb{R}^d$, we say $x : [0,T] \to \mathbb{R}^d$ is a solution of

$$\text{(ODE)} \begin{cases} \dot{x}(t) = v(x(t),t) \\ x(0) = x_0 \end{cases}$$

if

(i) x is abs cts on $[0,T]$

(ii) $\underbrace{x(t) = x_0 + \int_0^t v(x(s),s) \, ds}_{\text{hence (ODE) holds for a.e. } t \in [0,T]} \quad \forall t \in [0,T]$.

Recall:

Def: $f : [a,b] \to \mathbb{R}$ is absolutely continuous if

- $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for any $\{(a_i, b_i)\}_{i=1}^{n} \subseteq [a,b]$ disjoint
  $\sum_i^n |b_i - a_i| < \delta \Rightarrow \sum_i^n |f(b_i) - f(a_i)| < \varepsilon$

$\Updownarrow$

- $\exists\, g \in L^1([a,b])$ s.t.
  $(*)$ $f(t) - f(s) = \int_s^t g(r)\,dr \quad \forall s,t \in [a,b]$

$\Updownarrow$

- f is differentiable a.e. on $[a,b]$, $f' \in L^1([a,b])$ and $(*)$ holds for $g = f'$.

(See Folland 3.35)

Fix $v(x,t)$. Suppose that a soln of (ODE) exists for all $x_0 \in \mathbb{R}^d$. In this case, we may consider the flow map induced by v,

$$X_t(y) := x(t), \text{ where } x(t) \text{ solves (ODE) w/ i.c. } x_0 = y$$

Then, for all $t \in [0,T]$, $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is well defined.

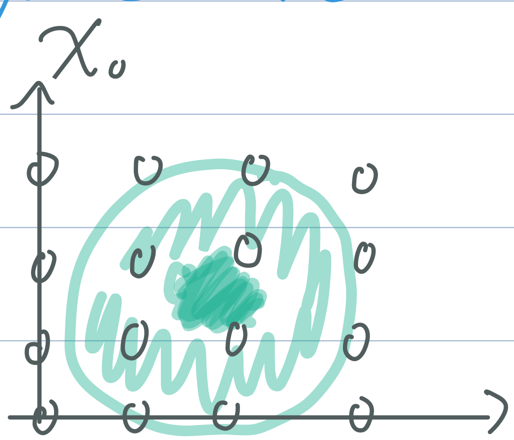
We now show that pushing forward a measure by this flow map gives a soln of (CE) with velocity v. This is the Lagrangian perspective.

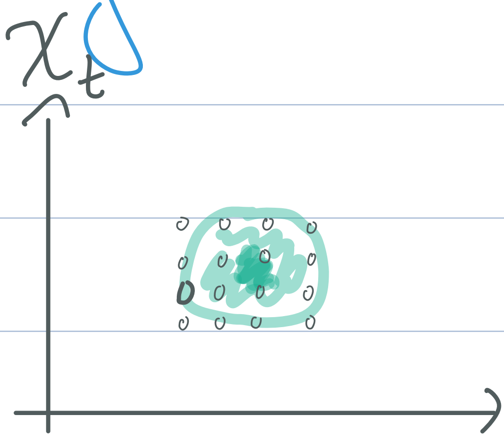
Prop: Fix $v(x,t)$, $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Suppose that solns of (ODE) exist for μ_0-a.e. initial cond $x_0 \in \mathbb{R}^d$ on $[0,T]$. Let $X_t : \mathbb{R}^d \to \mathbb{R}^d$ be the corresponding flow map, which is defined μ_0-a.e.

Define $\mu_t := X_t \# \mu_0$. Then $\{\mu_t\}_{t \in [0,T]}$ is a Borel family.

Furthermore if $\int_0^T \int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x) \, dt < +\infty$,

- $\mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ $\forall t \in [0,T]$
- μ is a weak soln of (CE) on $\mathbb{R}^d \times [0,T]$.





Pf:

- $t \mapsto \mu_t$ is narrowly cts, so $\{\mu_t\}_{t\in[0,T]}$ is a Borel family.

- Note that, by Jensen's ineq.

$$\left(\frac{1}{T}\int_0^T \int_{\mathbb{R}^d} |v(X_t(x),t)|\,d\mu_0(x)\,dt\right)^2$$

$$\parallel$$

$$\left(\frac{1}{T}\int_0^T \int_{\mathbb{R}^d} |v(x,t)|\,d\mu_t(x)\,dt\right)^2$$

$$\leq \frac{1}{T}\int_0^T \left(\int_{\mathbb{R}^d} |v(x,t)|\,d\mu_t(x)\right)^2 dt$$

$$\leq \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} |v(x,t)|^2\,d\mu_t(x)\,dt < +\infty$$

This shows (ii) in defn of (CE) soln holds

Pf: First, we show (i) in defn of weak soln. Fix $f \in C_c^\infty(\mathbb{R}^d)$. For $0 \le s < t \le T$,

$$\int_{\mathbb{R}^d} f \, d\mu_t - \int_{\mathbb{R}^d} f \, d\mu_s = \int_{\mathbb{R}^d} f(X_t(x)) - f(X_s(x)) \, d\mu_0(x)$$

- For x fixed, $t \mapsto X_t(x)$ is abs cts on $[0,T]$. Since f is Lipschitz, $t \mapsto f(X_t(x))$ is abs cts, so
$$f(X_t(x)) - f(X_s(x)) = \int_s^t \frac{d}{dr} f(X_r(x)) \, dr.$$

- For fixed x, $r \mapsto X_r(x)$ is diff for a.e. $r \in [0,T]$, for any such r,
$$\lim_{h \to 0} \frac{1}{h}\left(f(X_{r+h}(x)) - f(X_r(x))\right)$$
$$= \lim_{h \to 0} \underbrace{\frac{1}{h}\left(f(X_{r+h}(x)) - f(X_r(x) + h\dot{X}_r(x))\right)}_{\mathrm{I}}$$
$$+ \frac{1}{h}\left(f(X_r(x) + h\dot{X}_r(x)) - f(X_r(x))\right) = (*)$$

$\mathrm{I} \le \|f\|_{Lip} \left| \int_r^{r+h} \dot{X}(s) \, ds - \dot{X}_r(x) \right| \xrightarrow{h \to 0} 0$

by Lebesgue's diff theorem.

$(*) = \nabla f(X_r(x)) \cdot \dot{X}_r(x)$

Combining these...

$$\int_{\mathbb{R}^d} f \, d\mu_t - \int_{\mathbb{R}^d} f \, d\mu_s$$

$$= \int_{\mathbb{R}^d} \int_s^t \nabla f(X_r(x)) \cdot \overbrace{\dot{X}_r(x)}^{v(X_r(x),\,r)} \, dr \, d\mu_0(x)$$

Fubini (see ☼)

$$= \int_s^t \overbrace{\int_{\mathbb{R}^d} \nabla f(x) \cdot v(x,r) \, d\mu_r(x)}^{\in L^1([0,T],\,dr)} \, dr$$

Thus condition (i) of (CE) holds.

Finally, to see $\mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ $\forall t \in [0,T]$, by defn of ODE,

$$|X_t(x)| \leq |x| + \int_0^t |v(X_s(x), s)| \, ds$$

$\int |x|^2 d\mu_t(x)$

$= \int |X_t(x)|^2 d\mu_0(x)$

$\leq 2\int |x|^2 d\mu_0(x) + 2\int_{\mathbb{R}^d} \int_0^t |v(X_s(x),s)|^2 ds\, d\mu_0$

$= 2M_2(\mu_0) + 2\int_0^t \int_{\mathbb{R}^d} |v(x,s)|^2 d\mu_s(x)\, ds$

$< +\infty$ $\square$

Now: use this Lagrangian perspective to connect geodesics in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ to solns of (CE).

Def: Given a metric space (X,d), $x: [0,1] \to X$ is a (constant speed) geodesic if, $\forall\, s,t \in [0,1]$,

$$d(x(t),x(s)) = |t-s|\, d(x(0),x(1))$$

<u>Prop</u>: Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu_0, \mu_1 \ll \mathcal{L}^d$ (*), there exists a constant speed geodesic μ between them.

Furthermore, there exists a velocity $v(x,t)$ s.t. (μ, v) solves (CE) and

$$\left(\int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x) \right)^{1/2} \equiv W_2(\mu_0, \mu_1) \ \forall t \in [0,1]$$

Rmk: The prop continues to hold even without (*).

The proof relies on the following lemma...

<u>Lemma</u>: Suppose $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu, \nu \ll \mathcal{L}^d$.
Let T be the OT map from μ to ν.
Let $\tilde{T}$ be the OT map from ν to μ.
Then $T \circ \tilde{T} = id$ ν-a.e., $\tilde{T} \circ T = id$ μ-a.e.,
so T is left invertible μ-a.e. with inverse $\tilde{T}$.

<u>Pf</u>: Since, by Brenier's Thm, OT plans are unique,

$$\underbrace{(id \times T)\#\mu}_{\gamma} = \underbrace{(\tilde{T} \times id)\#\nu}_{\gamma'}$$

Thus,

$$\int |y - T \circ \tilde{T}(y)| \, d\nu(y)$$

$$= \int |x^2 - T(x^2)| \, d\gamma'(x^1, x^2)$$

$$= \int |x^2 - T(x^1)| \, d\gamma(x^1, x^2)$$

$$= \int |T(x) - T(x)| \, d\mu(x) = 0$$

Hence $y = T \circ \tilde{T}(y)$ ν-a.e.

The rest of the lemma follows symmetrically. □

Now, we prove the proposition.

Pf: Since $\mu_0, \mu_1 \ll \mathcal{L}^d$, $\exists$ OT map T from μ_0 to μ_1.
Define

$$T_t(x) := (1-t)x + tT(x)$$
$$\mu_t := T_t \# \mu_0$$

We want to use T_t to define $v(x,t)$, but to do this, we will need T_t to have a left inverse μ_t-a.e.. By the previous lemma, this will hold, provided that we can show $\mu_t \ll \mathcal{L}^d$.