Lecture 17.

Reminders
- Solutions for 2-3 exercises
- Revise article by March 7th
- No class on Thursday, Mar 6th
- Makeup Lecture, Friday, March 14, 9:30-10:45am SH6635

Recall:

Def: Given a metric space (X, d), $x: [0,1] \to X$ is a (constant speed) geodesic if, $\forall$ $s,t \in [0,1]$,

$$d(x(t), x(s)) = |t-s| \, d(x(0), x(1))$$

Prop: Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu_0, \mu_1 \ll \mathcal{L}^d$, $(\cancel{*})$ there exists a constant speed geodesic μ between them.

Furthermore, there exists a velocity $v(x,t)$ s.t. (μ, v) solves (CE) and

$$\left(\int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x) \right)^{1/2} \equiv W_2(\mu_0, \mu_1) \quad \forall t \in [0,1]$$

Rmk: The prop continues to hold even without $(*)$.

The proof relies on the following lemma...

<u>Lemma</u>: Suppose $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu, \nu \ll \mathcal{L}^d$.
Let T be the OT map from μ to ν.
Let $\tilde{T}$ be the OT map from ν to μ.
Then $T \circ \tilde{T} = id$ ν-a.e., $\tilde{T} \circ T = id$ μ-a.e.,
so T is left invertible μ-a.e. with inverse $\tilde{T}$.

Now, we prove the proposition.

Pf: Since $\mu_0, \mu_1 \ll \mathcal{L}^d$, $\exists$ OT map T from μ_0 to μ_1.
Define

$$T_t(x) := (1-t)x + tT(x)$$

$$\mu_t := T_t \# \mu_0$$

We want to use T_t to define $v(x,t)$, but to do this, we will need T_t to have a left inverse μ_t-a.e.. By the previous lemma, this will hold, provided that we can show $\mu_t \ll \mathcal{L}^d$.

Recall that, by Brenier's Thm, $\exists\ \varphi \in L^1(\mu_0)$ proper, convex, lsc s.t.

$$T(x) = \begin{cases} \nabla \varphi(x) & \text{for all } x \in A \\ 0 & x \in A^c \end{cases}$$

where $\mu_0(A^c) = 0$

Since φ is convex, by Exercise 34,

$\langle \nabla\varphi(x) - \nabla\varphi(y), x-y \rangle \geq 0$ for $x, y \in A$.

Thus,

$\langle T_t(x) - T_t(y), x-y \rangle \geq (1-t)^2 |x-y|^2$, $x, y \in A$

$\Downarrow$

$|T_t(x) - T_t(y)| \geq (1-t)^2 |x-y|$

$\Downarrow$

T_t is injective on A

$\Downarrow$

T_t is invertible onto $T_t(A)$, with Lipschitz cts inverse, when $t<1$.

Fix $B \in \mathcal{B}(\mathbb{R}^d)$ with $\mathcal{L}^d(B) = 0$.

Then, $\forall \varepsilon > 0$, $\exists \{I_k\}_{k=1}^{\infty}$ s.t.

$B \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_k \mathcal{L}^d(I_k) < \varepsilon$.

Thus, for $S = B \cap T_t(A)$,

$$\mathcal{L}^d(T_t^{-1}(S)) \leq \mathcal{L}^d\left(T_t^{-1}\left(\bigcup_{k=1}^{\infty} I_k\right)\right)$$

$$= \mathcal{L}^d\left(\bigcup_{k=1}^{\infty} T_t^{-1}(I_k)\right)$$

$$\leq \sum_k \mathcal{L}^d(T_t^{-1}(I_k))$$

$$\leq \sum_k \frac{1}{(1-t)^{2d}} \mathcal{L}(I_k)$$

$$\leq \frac{\varepsilon}{(1-t)^{2d}}$$

Since $\varepsilon > 0$ was arbitrary, $\mathcal{L}^d(T_t^{-1}(S)) = 0$.

Then $\mu_0(T_t^{-1}(S)) = 0$, so

$$0 = \mu_0(T_t^{-1}(S)) = \mu_0(T_t^{-1}(B \cap T_t(A)))$$

$$= \mu_0(T_t^{-1}(B) \cap A) = \underset{\mu_t(B)}{\underset{\shortparallel}{\mu_0(T_t^{-1}(B))}}.$$

Thus $\mu_t \ll \mathcal{L}^d$.

By prev prop, we know T_t invertible μ_t-a.e. and $T_t^{-1} \# \mu_t = \mu_0$.

Now, we will show μ_t is a geo from μ_0 to μ_1. Note that $(T_t \times T_s) \# \mu_0 \in \Gamma(\mu_t, \mu_s)$. Thus

$$
(*) \quad \left\{ \begin{aligned}
W_2^2(\mu_t, \mu_s) &\leq \int |x-y|^2 d((T_t \times T_s) \# \mu_0)(x,y) \\
&= \int |T_t(x) - T_s(x)|^2 d\mu_0(x) \\
&= |t-s|^2 \int |x - T(x)|^2 d\mu_0(x) \\
&= |t-s|^2 W_2^2(\mu_0, \mu_1)
\end{aligned} \right.
$$

For the other direction of the inequality, note that, for $t \geq s$,

$$
\begin{aligned}
W_2(\mu_0,\mu_1) &\le W_2(\mu_0,\mu_s) + W_2(\mu_s,\mu_t) + W_2(\mu_t,\mu_1) \\
(*)\quad &\le (s + (t-s) + (1-t))\, W_2(\mu_0,\mu_1) \\
&= W_2(\mu_0,\mu_1)
\end{aligned}
$$

Thus, equality holds throughout, and μ_t is a geodesic.

It remains to show μ_t solves (CTY) for some velocity. Define,

$$v(x,t) = T \circ T_t^{-1}(x) - T_t^{-1}(x)$$

for μ_t-a.e. x.

First, we compute the kinetic energy...

$$
\begin{aligned}
\int |v(x,t)|^2 \, d\mu_t(x) &= \int |v(T_t(x),t)|^2 \, d\mu_0(x) \\
&= \int |T(x) - x|^2 \, d\mu_0(x) \\
&= W_2^2(\mu_0, \mu_1)
\end{aligned}
$$

Lastly, to see (μ, v) solves (CTY), note that by defn, for μ_0-a.e. x,

$$
\begin{aligned}
\frac{d}{dt} T_t(x) &= T(x) - x \\
&= v(T_t(x), t)
\end{aligned}
$$

Thus $\mu_t = T_t \# \mu_0$ solves (CTY) with velocity v. $\square$

Now, we will work to show that <u>all</u> (reasonably) regular curves in $(\mathbb{P}_2(\mathbb{R}^d), W_2)$ solve (CTY).

What do we mean by "reasonably regular"?

Suppose (X,d) is a complete metric space.

<u>Def</u>: Fix $p\geq 1$, $0\leq a<b$. Then $x:(a,b)\to X$ is <u>p-absolutely continuous</u>, denoted $x\in AC^p(a,b;X)$, if $\exists\ \underbrace{g\in L^p((a,b))}_{(*)}$ s.t.

$$d(x(t),x(s))\leq\int_s^t g(r)\,dr,\ \forall\ a<s\leq t<b. \quad (*)$$

<u>Rmk</u>: For $q\leq p$, $AC^p(a,b;X)\subseteq AC^q(a,b;X)$

For $p=+\infty$, this is Lipschitz cty.

For $p\geq 1$, $AC^p(a,b;X)\subseteq C([a,b];X)$

Def: The metric derivative of $x:(a,b)\to X$ is

$$|x'|(t) := \lim_{h\to 0} \frac{d(x(t+h), x(t))}{|h|}$$

Prop: For any $x \in AC^p(a,b;X)$,
(i) $|x'|(t)$ exists for $\mathcal{L}^1$-a.e. $t\in(a,b)$
(ii) $g(t) := |x'|(t)$ is admissible in (*).
(iii) $|x'|(t) \leq g(t)$ $\mathcal{L}^1$-a.e. t for all g satisfying (*).

Pf: Since $x:(a,b)\to X$ is cts and (a,b) is separable, $x((a,b))$ is separable. Let $\{y_n\}_{n=1}^{\infty}$ be a dense sequence and define

$$d_n(t) = d(y_n, x(t)).$$

First, we show (i).

By the reverse triangle ineq, for any g satisfying $(*)$, we have

$$\underbrace{|d_n(t)}_{= d(y_n, x(t))} - \overbrace{d_n(s)}^{= d(y_n, x(s))}| \le d(x(t), x(s)) \le \int_s^t g(r)\,dr$$

Thus, $t \mapsto d_n(t)$ is abs cts and

$$D(t) := \sup_{n \in \mathbb{N}} |d_n'(t)|$$

is well-defined $\mathcal{L}^1$-a.e. $t \in (a,b)$.

Thus, for $\mathcal{L}^1$-a.e. $t \in (a,b)$,

$$D(t) = \sup_{n \in \mathbb{N}} \liminf_{s \to t} \frac{|d_n(s) - d(t)|}{|s-t|}$$

$$\le \liminf_{s \to t} \frac{d(x(t), x(s))}{|s-t|} \le g(t) \quad \cancel{(**)}$$

On the other hand, by density of $\{y_n\}_{n=1}^{\infty}$,

$(*)$

$$
\begin{aligned}
d(x(t),x(s)) &= \sup_{n\in\mathbb{N}} |d_n(t)-d_n(s)| \\
&= \sup_{n\in\mathbb{N}} \left| \int_s^t d_n'(r)\,dr \right| \\
&\le \sup_{n\in\mathbb{N}} \int_s^t |d_n'(r)|\,dr \\
&\le \int_s^t D(r)\,dr
\end{aligned}
$$

Thus, for $\mathcal{L}^1$-a.e. t,

$$\limsup_{t\to s} \frac{d(x(s),x(t))}{|s-t|} \le D(t)$$

Thus, $|x'|(t)$ exists for $\mathcal{L}^1$-a.e. $t \in (a,b)$ and $|x'|(t) = D(t)$.

Part (ii) follows from $(**)$.
Part (iii) follows from $(***)$. $\square$

We will show that all abs cts curves in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ solve (CTY). To do this, a key quantity will be the <u>kinetic energy</u> of such a curve.

(Spoiler: kinetic energy will be metric derivative.)

Def: For $(r,x) \in \mathbb{R} \times \mathbb{R}^d$,

$$f_{\mathbb{B}}(r,x) = \begin{cases} \frac{1}{2}\frac{|x|^2}{r} & \text{if } r>0 \\ 0 & \text{if } r=x=0 \\ +\infty & \text{if } r=0, x\neq 0 \\ & \text{or } r<0 \end{cases}$$

Exercise 35:

For $(s,y) \in \mathbb{R} \times \mathbb{R}^d$, define

$$g_{\mathbb{B}}(s,y) := \chi_{\{s+\frac{1}{2}|y|^2 \leq 0\}}(s,y).$$

Then $g_{\mathbb{B}}^* = f_{\mathbb{B}}$ and $f_{\mathbb{B}}^* = g_{\mathbb{B}}$.

We will now use these to define a generalized notion of kinetic energy.

Prop: Given $\mu \in \mathcal{M}(\mathbb{R}^d)$, $m \in \mathcal{M}_s^d(\mathbb{R}^d)$ define

$$\mathcal{B}(\mu, m) := \sup\left\{ \int f \, d\mu + \int g \cdot dm \,:\, f \in C_b(\mathbb{R};\mathbb{R}),\ g \in C_b(\mathbb{R}^d, \mathbb{R}^d),\ f + \tfrac{1}{2}|g|^2 \leq 0 \right\}.$$

Then $\mathcal{B}(\mu, m)$ is convex and lsc wrt narrow conv.

Pf: Follows from Exercise 11.

μ_t, $\quad |\mu_t'| < +\infty \implies \mathcal{B}(\mu, m) < +\infty$

$$\Downarrow$$

$\partial_t \mu = -\nabla \cdot (v\mu) \qquad m = v\mu$