# Lecture 18

Reminders
- Solutions for 2-3 exercises
- Revise article by March $7^{th}$
- Thursday lecture begins 9:45am
- Makeup Lecture, Friday, March 14, 9:30-10:45am SH6635

Recall:

Now, we show that <u>all</u> reasonably regular curves in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ solve (CE).

What is meant by "reasonably regular"?

Suppose $(X, d)$ is a complete metric space.

**Def:** Fix $p \geq 1$, $0 \leq a < b$. Then $x : (a,b) \to \mathcal{X}$ is <u>$p$-absolutely continuous</u>, denoted $x \in AC^p(a,b;\mathcal{X})$, if $\exists\, g \in L^p((a,b))$ s.t.

$$\underbrace{d(x(t), x(s)) \leq \int_s^t g(r)\,dr,}_{(*)} \quad \forall\, a < s \leq t < b.$$

**Def:** The <u>metric</u> <u>derivative</u> of $x : (a,b) \to \mathcal{X}$ is

$$|x'|(t) := \lim_{h \to 0} \frac{d(x(t+h), x(t))}{|h|}$$

**Prop:** For any $x \in AC^p(a,b;\mathcal{X})$,

(i) $|x'|(t)$ exists for $\mathcal{L}^1$-a.e. $t \in (a,b)$.

(ii) $g(t) := |x'|(t)$ is admissible in $(*)$.

(iii) $|x'|(t) \leq g(t)$ $\mathcal{L}^1$-a.e. $t$, for all $g$ satisfying $(*)$.

We will show that all abs cts curves in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ solve (CE).

To do this, a key quantity will be the <u>kinetic energy</u> of such a curve.

(Spoiler: kinetic energy will be metric derivative.)

<u>Def</u>: For $(r,x) \in \mathbb{R} \times \mathbb{R}^d$,

$$f_B(r,x) = \begin{cases} \frac{1}{2}\frac{|x|^2}{r} & \text{if } r>0 \\ 0 & \text{if } r=x=0 \\ +\infty & \text{if } r=0, x\neq 0 \text{ or } r<0 \end{cases}$$

Exercise 35:

$$g_{\mathbb{B}}(s,y) := \chi_{\{s+\frac{1}{2}|y|^2 \leq 0\}}(s,y)$$

satisfies $g_{\mathbb{B}}^{*} = f_{\mathbb{B}}$ and $f_{\mathbb{B}}^{*} = g_{\mathbb{B}}$.

<u>Prop</u>: Given $\mu \in \mathcal{M}(\mathbb{R}^d)$, $m \in \mathcal{M}_s^d(\mathbb{R}^d)$ define

$$\underbrace{\mathbb{B}(\mu,m)}_{\text{Kinetic Energy}} := \sup\Big\{\int f\,d\mu + \int g\cdot dm \;:\; f \in C_b(\mathbb{R};\mathbb{R}),\ g \in C_b(\mathbb{R}^d,\mathbb{R}^d),\ f+\tfrac{1}{2}|g|^2 \leq 0\Big\}.$$

Then $\mathbb{B}(\mu,m)$ is convex and lsc wrt narrow conv.

Now: properties of $\mathbb{B}$ and why we call it kinetic energy.

<u>Prop</u>:

(i) $\mathbb{B}(\mu,m) = \overbrace{\sup_{\substack{f\in L^\infty(\mathbb{R};\mathbb{R}),\ g\in L^\infty(\mathbb{R}^d,\mathbb{R}^d)\\ f+\frac{1}{2}|g|^2\leq 0}} \left\{ \int f\,d\mu + \int g\cdot dm \right\}}^{(*)}$.

(ii) Suppose $\mu, m \ll \omega$, where $\omega$ is a $\sigma$-finite Borel measure on $\mathbb{R}^d$. Then

$$\mathbb{B}(\mu,m) = \int f_{\mathbb{B}}\left(\frac{d\mu}{d\omega}, \frac{dm}{d\omega}\right) d\omega$$

(iii) $$\mathbb{B}(\mu,m) = \begin{cases} \frac{1}{2}\int |v|^2 d\mu & \boxed{\begin{array}{l}\text{if } m \ll \mu \\ dm = v\,d\mu\end{array}} \\ +\infty & \text{otherwise} \end{cases}$$

Our proof leverages the following classical result, Rockafellar "Integrals Which are Convex Functions."

$\frac{1}{p} + \frac{1}{p'} = 1$

<u>Thm</u>: Given a $\sigma$-finite Borel measure $\omega$ on $\mathbb{R}^d$, $p \geq 1$, and $F: \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lsc, if $F(f)$ is integrable for some $f \in L^p(\omega)$ and $F^*(g)$ is integrable for some $g \in L^{p'}(\omega)$ then

$\mathcal{F}: L^p(\omega) \to \mathbb{R} \cup \{+\infty\}$, $\mathcal{F}(f) := \int F(f)\, d\omega$

$\mathcal{G}: L^{p'}(\omega) \to \mathbb{R} \cup \{+\infty\}$, $\mathcal{G}(g) := \int F^*(g)\, d\omega$

are well-defined, proper, convex functions and $\mathcal{G} = \mathcal{F}^*$, $\mathcal{F} = \mathcal{G}^*$.

Pf of Prop:

First, we will show (i). By def, $\mathcal{B}(\mu, m) \leq (*)$. It remains to show the opposite inequality.

Note that

$$(*) = \sup_{n \in \mathbb{N}} \sup_{\substack{g \in L^\infty(\mathbb{R}^d; \mathbb{R}^d) \\ \|g\|_\infty \leq n}} \left\{ -\int \frac{|g|^2}{2} d\mu + \int g \cdot dm \right\}$$

For any $g_n \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$, $\|g_n\|_\infty \leq n$...

- Lusin's Thm guarantees that, $\forall \varepsilon > 0$, $\exists E_{n,\varepsilon} \subset\subset \mathbb{R}^d$ s.t. $(\mu + |m|)(E_{n,\varepsilon}') < \frac{\varepsilon}{n^2}$ and $g_n|_{E_{n,\varepsilon}}$ is cts.
- Tietze extension guarantees $\exists g_{n,\varepsilon} \in C_b(\mathbb{R}^d; \mathbb{R}^d)$ s.t. $\|g_{n,\varepsilon}\|_\infty \leq \|g_n\|_\infty$ and $g_{n,\varepsilon} = g_n$ on $\bar{E}_{n,\varepsilon}$.

Thus

$$-\int \frac{|g_n|^2}{2}\,d\mu + \int g_n \cdot dm$$

$$= -\int \frac{|g_{n,\varepsilon}|^2}{2}\,d\mu + \int g_{n,\varepsilon}\cdot dm + \underbrace{(\mu + |m|)(E_{n,\varepsilon}^c)(n^2+2n)}_{\leq 2\varepsilon}$$

Therefore,

$$\sup_{\substack{g \in L^\infty(\mathbb{R}^d;\mathbb{R}^d) \\ \|g\|_\infty \leq n}} -\int \frac{|g|^2}{2}\,d\mu + \int g\cdot dm \leq \sup_{\substack{g \in C_b(\mathbb{R}^d;\mathbb{R}^d) \\ \|g\|_\infty \leq n}} -\int \frac{|g|^2}{2}\,d\mu + \int g \cdot dm + 2\varepsilon$$

$\varepsilon > 0$ arbitrary, send $\varepsilon \to 0$

Taking sup over $n \in \mathbb{N}$ gives

$(*) \leq \mathcal{B}(\mu, m)$.

We now show part (ii) via Rockafellar's Thm.

Define

$\mathcal{F}: \overbrace{L^1(\omega)}^{L^1_\omega(\mathbb{R}^d;\mathbb{R}^{d+1})} \to \mathbb{R}\cup\{+\infty\}$, $\mathcal{F}(f) = \int f_{\mathbb{B}}(\overset{\varphi}{\cancel{f}})\,d\omega$.

If $A \in \mathcal{B}(\mathbb{R}^d)$ s.t. $\omega(A) < +\infty$, then

$$\mathcal{F}(\mathbb{1}_A) = \frac{1}{2}\int_A d\omega = \frac{1}{2}\omega(A) < +\infty.$$

Define

$\mathcal{G}: L^\infty(\omega) \to \mathbb{R}\cup\{+\infty\}$, $\mathcal{G}(g) = \int f^*_{\mathbb{B}}(\overset{\psi}{\cancel{g}})\,d\omega$

Note that $\mathcal{G}(0) = \int 0 \, d\omega = 0$.

Thus, Rockafellar's Thm ensures

$$\mathbb{B}(\mu, m) = \sup_{\substack{f \in L^\infty(\mathbb{R};\mathbb{R}),\ g \in L^\infty(\mathbb{R}^d, \mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0}} \left\{ \int f \, d\mu + \int g \cdot dm \right\}$$

$$= \sup_{\substack{f \in L^\infty_w(\mathbb{R};\mathbb{R}),\ g \in L^\infty_w(\mathbb{R}^d, \mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0 \ w\text{-a.e.}}} \left\{ \int \left( f \frac{d\mu}{dw} + g \cdot \frac{dm}{dw} \right) dw \right\}$$

$$= \sup_{f \in L^\infty_w(\mathbb{R};\mathbb{R}),\ g \in L^\infty_w(\mathbb{R}^d, \mathbb{R}^d)} \left\{ \int \left( f \frac{d\mu}{dw} + g \cdot \frac{dm}{dw} \right) dw - \overbrace{\int g_{\mathbb{B}}(f,g)\, dw}^{\mathcal{G}(f,g)} \right\}$$

$$= \mathcal{G}^*\left( \frac{d\mu}{dw}, \frac{dm}{dw} \right)$$

$$= \mathcal{F}\left( \frac{d\mu}{dw}, \frac{dm}{dw} \right)$$

$$= \int f_{\mathbb{B}}\left( \frac{d\mu}{dw}, \frac{dm}{dw} \right) dw$$

Finally, we show part (iii).

If $\underbrace{m \ll \mu}_{dm = v\,d\mu}$, then applying part (ii) with $w := \mu$ gives

$$\mathbb{B}(\mu, m) = \int f_{\mathbb{B}}(1, v)\, d\mu = \frac{1}{2}\int |v|^2 d\mu.$$

OTOH, if $m \not\ll \mu$, then $\exists\, A \in \mathcal{B}(\mathbb{R}^d)$ s.t. $\mu(A) = 0$ but $m(A) \neq 0$.

Define $f_n := -\frac{n^2}{2} 1_A$, $g_n := n \frac{m(A)}{|m(A)|} 1_A$ then $f_n + \frac{1}{2}|g_n|^2 \leq 0$, so they satisfy constraints in defn of $\mathbb{B}$ and

$$\mathbb{B}(\mu, m) \geq \sup_n \underbrace{\int f_n\, d\mu}_{=0} + \underbrace{\int g_n \cdot dm}_{= n|m(A)|} = +\infty. \square$$

Theorem (characterization of $AC^2$ curves and solns of (CE))

(i) Suppose $\mu \in AC^2(0,T; \mathcal{P}_2(\mathbb{R}^d))$.

Then $\exists\ v$ s.t. $(\mu, v)$ solve (CE) and

$$\left(\int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x)\right)^{1/2} \leq |\mu'|(t), \text{ a.e. } t$$

(ii) Suppose $(\mu, v)$ solve (CE) and

$$\int_0^T \int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x)\, dt < +\infty.$$

Then $\mu \in AC^2(0,T; \mathcal{P}_2(\mathbb{R}^d))$ and

$$|\mu'|(t) \leq \left(\int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x)\right)^{1/2}, \text{ a.e. } t.$$

Rmk: If the result holds for $T=1$, then, by reparametrizing in time, it holds for all $T>0$.

We will prove this theorem under the simplifying assumption that $\exists\ R>0$ s.t. $\mu_t(B_R^c) \equiv 0$ $\forall\, t \in [0,1]$.

Our proof of (i) relies on a lemma:

Lemma: Given $\{m_k\}_{k\in\mathbb{N}} \subseteq \mathcal{M}_s^d(X)$ on a Polish space $X$ satisfying

- $\sup_k |m_k|(X) < +\infty$
- $\{|m_k|\}_{k\in\mathbb{N}}$ is tight

then $\{m_k\}_{k\in\mathbb{N}}$ is narrowly relatively compact.

Pf: Exercise 36.

Pf of Thm: We begin with (i).

Fix $\mu \in AC^2(0,T;\mathcal{P}_2(\mathbb{R}^d))$.
For $k \in \mathbb{N}$, consider the "discrete time" sequence

$$\mu_{0/k},\ \mu_{1/k},\ \dots,\ \mu_{i/k},\ \dots,\ \mu_{k/k}$$

By prev prop, for all $i=0,\dots,k-1$ there exists a geo $\mu_i^k(t)$ from $\mu_{i/k}$ to $\mu_{i+1/k}$ and $\exists$ velocity $v_i^k$ s.t. $(\mu_i^k, v_i^k)$ solve (CE) and

$$\left(\int |v_i^k(x,t)|^2 d\mu_{i,t}^k(x)\right)^{1/2} = W_2(\mu_{i/k}, \mu_{i+1/k})$$

$\forall\, t \in [0,1]$.

Now, chain these geodesics together, defining

$$\mu^k(t) := \mu_i^k(tk-i) \text{ for } t \in [i/k, i+1/k)$$
$$v^k(t) := v_i^k(tk-i) \cdot k$$

Furthermore, for $t \in [i/k, i+1/k)$

$$\int |v^k(x,t)|^2 d\mu_t^k(x) = k^2 W_2^2(\mu_{i/k}, \mu_{i+1/k})$$

$$\leq \left(k \int_{i/k}^{i+1/k} |\mu'|(s)ds\right)^2$$

$$\leq k \int_{i/k}^{i+1/k} |\mu'|^2(s)\, ds$$

Thus, for all $k \in \mathbb{N}$,

- $\int_0^1 \int_{\mathbb{R}^d} |v^k(x,t)| d\mu_t^k(x)\, dt < +\infty$

- $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$, $t \mapsto \int_{\mathbb{R}^d} \varphi\, d\mu_t^k$ is abs cts

- $\partial_t \mu^k + \nabla\cdot(\mu^k v^k) = 0$ holds in weak sense

Next time: identify a limit of $\mu^k_t$ as $k \to +\infty$; show that limit satisfies (CE); show that limit coincides w/ original curve $\mu$.