Lecture 19

Reminders
- Solutions for 2-3 exercises
- Revise article by March 7th
- Makeup Lecture, Friday, March 14, 9:30-10:45am SH6635

Recall:

Prop:

(i) $\mathcal{B}(\mu,m) = \sup\{\int f\,d\mu + \int g\cdot dm\}$.
$f\in L^\infty(\mathbb{R};\mathbb{R})$, $g\in L^\infty(\mathbb{R}^d,\mathbb{R}^d)$
$f+\frac{1}{2}|g|^2\le 0$

(ii) Suppose $\mu, m \ll \omega$, where ω is a σ-finite Borel measure on $\mathbb{R}^d$. Then

$$\mathcal{B}(\mu,m) = \int f_{\mathcal{B}}\left(\frac{d\mu}{d\omega},\frac{dm}{d\omega}\right)d\omega$$

(iii) $$\mathcal{B}(\mu,m) = \begin{cases} \frac{1}{2}\int |v|^2 d\mu & \boxed{\text{if } m\ll\mu,\ dm = v\,d\mu} \\ +\infty & \text{otherwise} \end{cases}$$

Theorem (characterization of AC^2 curves and solns of (CE))

(i) Suppose $\mu \in AC^2(0,T;\mathcal{P}_2(\mathbb{R}^d))$.
Then $\exists\, v$ s.t. (μ,v) solve (CE) and

$$\left(\int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x)\right)^{1/2} \leq |\mu'|(t), \text{ a.e. } t$$

(ii) Suppose (μ,v) solve (CE) and

$$\int_0^T \int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x)\,dt < +\infty.$$

Then $\mu \in AC^2(0,T;\mathcal{P}_2(\mathbb{R}^d))$ and

$$|\mu'|(t) \leq \left(\int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x)\right)^{1/2}, \text{ a.e. } t.$$

Rmk: If the result holds for $T=1$, then, by reparametrizing in time, it holds for all $T>0$.

We will prove this theorem under the simplifying assumption that $\exists\, R>0$ s.t. $\mu_t(B_R^c) \equiv 0$ $\forall\, t \in [0,1]$.

Our proof of (i) relies on a lemma:

<u>Lemma</u>: Given $\{\sigma_k\}_{k\in\mathbb{N}} \subseteq \mathcal{M}(X)$ on a Polish space X satisfying

- $\sup_k \sigma_k(X) < +\infty$
- $\{\sigma_k\}_{k\in\mathbb{N}}$ is tight

then $\{\sigma_k\}_{k\in\mathbb{N}}$ is relatively narrowly cpt.

<u>Lemma</u>: Given $\{\sigma_k\}_{k\in\mathbb{N}} \subseteq \mathcal{M}(X)$ on a Polish space X s.t. $\sigma_k \to \sigma$ narrowly, for any closed set $C \subseteq X$, $\sigma_k|_C \to \sigma|_C$ narrowly.

Pf: Exercise 37.

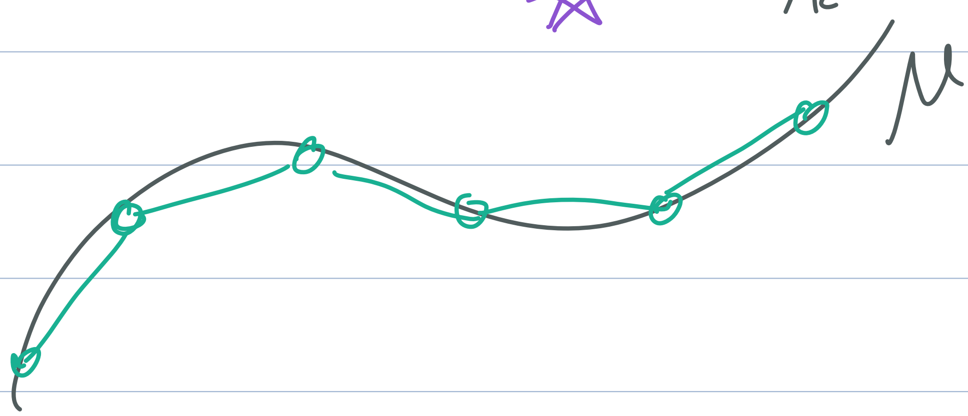
Pf of Thm: We begin with (i).

Fix $\mu \in AC^2(0,T;\mathcal{P}_2(\mathbb{R}^d))$.
For $k \in \mathbb{N}$, consider the "discrete time" sequence

$$\mu_{0/k}, \mu_{1/k}, \dots, \mu_{i/k}, \dots, \mu_{k/k}$$

Now, chain these together w/ geodesics (μ^k, v^k).

$$\int |v^k(x,t)|^2 d\mu^k_t(x) \leq \underbrace{k \int_{i/k}^{i+1/k} |\mu'|^2(s)\, ds}_{f_k(t)} \quad \star$$



Thus, for all $k \in \mathbb{N}$,

- $\int_0^1 \int_{\mathbb{R}^d} |v^k(x,t)| \, d\mu_t^k(x) \, dt < +\infty$
- $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$, $t \mapsto \int_{\mathbb{R}^d} \varphi \, d\mu_t^k$ is abs cts
- $\partial_t \mu^k + \nabla \cdot (\mu^k v^k) = 0$ holds in weak sense

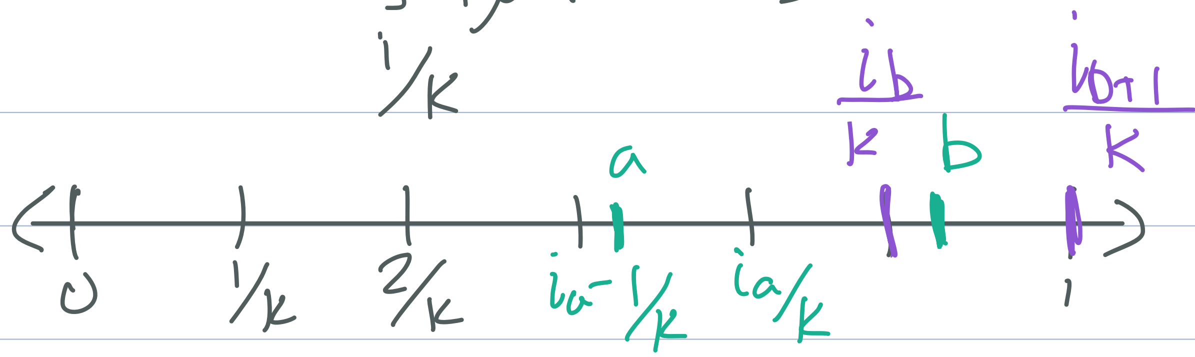
Next time: identify a limit of μ_t^k as $k \to +\infty$; show that limit satisfies (CE); show that limit coincides w/ original curve μ.

Define $f_k : [0,1] \to [0,+\infty]$ by

$$f_k(t) := k \int_{i/k}^{(i+1)/k} |u'|^2(s)\,ds \quad \text{for } t \in \left[\frac{i}{k}, \frac{i+1}{k}\right).$$

Note that, $\forall\, i,k$,

$$\int_{i/k}^{(i+1)/k} f_k(t)\,dt = \int_{i/k}^{(i+1)/k} |u'|^2(s)\,ds$$



Thus, for $[a,b] \subseteq [0,1]$,

$$\int_a^b f_k(t)\,dt \le \int_{\left[\frac{i_a-1}{k}, \frac{i_a}{k}\right] \cup \left[\frac{i_b}{k}, \frac{i_b+1}{k}\right]} |u'|^2(s)\,ds + \int_a^b |u'|^2(s)\,ds$$

Since $|u'|^2(s) \in L^1([0,1])$,

we conclude

$$\limsup_{k\to\infty} \int_a^b f_k(t)\,dt \le \int_a^b |\mu'|^2(s)\,ds$$

Therefore, by ☆,

$$\limsup_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} |v^k(x,t)|^2\,d\mu_t^k(x)\,dt \le \int_a^b |\mu'|^2(s)\,ds$$

By Hölder's inequality,

(✱)
$$\limsup_{k\to\infty} \int_0^1 \int_{\mathbb{R}^d} |v^k(x,t)|\,d\mu_t^k(x)\,dt$$

$$\le \limsup_{k\to\infty} \int_0^1 \left(\int |v^k(x,t)|^2\,d\mu_t^k\right)^{1/2} \cdot 1\,dt$$

$$\le \limsup_{k\to\infty} \left(\int_0^1 \int |v^k(x,t)|^2\,d\mu_t^k\,dt\right)^{1/2} < +\infty$$

Define $m^k \in \mathcal{M}_s^d(\mathbb{R}^d \times [0,1])$ by

$$m^k = v_t^k\,d\mu_t^k\,dt$$

Since μ_t is cptly supported in B_R, so is μ_t^k, so m^k is also cptly supported in $B_R \times [0,T]$, hence the positive and negative part of each component is tight.

Estimate $(*)$ shows that pos & neg part of each component has bounded mass.

Thus, up to a subsequence, $\exists$ $m \in \mathcal{M}_s^d(\mathbb{R}^d \times [0,1])$ s.t. $m_k \to m$ narrowly.

Likewise, for any $b \in [0,1]$,

$$W_2(\mu_b^k, \mu_b)$$
$$\leq W_2(\mu_b^k, \mu_{i_b/k}) + W_2(\mu_{i_b/k}, \mu_b)$$

↙ geo from $\mu_{i_b/k}$ to $\mu_{i_b+1/k}$ b

$$\leq W_2(\mu_i^k(\underset{\text{sped up time}}{bk - i_b}), \mu_{i_b/k}) + \int_{i_{b}/k}^{i_b+1/k} |\mu'|(s)ds$$

$$\leq |bk-i_b| W_2(\mu_{i_b/k}, \mu_{i_b+1/k}) + \int_{i_b/k}^{i_b/k} |\mu'|(s)ds$$

$$\leq (|bk-i_b|+1) \int_{i_b/k}^{i_b+1/k} |\mu'|(s) \cdot 1 ds$$

Hölder

$$\leq \underbrace{(|bk-i_b|+1)}_{\leq 1} \underbrace{\sqrt{1/k}}_{k \to +\infty \to 0} \underbrace{\left(\int_{i_b/k}^{i_b+1/k} |\mu'|^2(s)ds\right)^{1/2}}_{\leq \int_0^1 |\mu'|^2(s)ds}$$

$$\left|b - \frac{i_b}{k}\right| \leq \frac{1}{k} \Rightarrow |bk - i_b| \leq 1$$

Thus $\mu_b^k \to \mu_b$ narrowly $\forall b \in [0,1]$

Hence, by DCT $\mu_t^k dt \to \mu_t dt$ narrowly.

Thus, $\forall \varphi \in C_c^\infty(\mathbb{R}^d \times [0,1])$, $\forall k \in \mathbb{N}$

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(x,t) \underline{d\mu_t^k dt} + \int_0^1 \int_{\mathbb{R}^d} \underline{v^k(x,t)} \cdot \nabla\varphi(x,t) \underline{d\mu_t^k(x) dt} = 0$$

$$\Big\downarrow k \to +\infty$$

$$\int_{\mathbb{R}^d} \partial_t \varphi(x,t) \underline{d\mu_t dt} + \int_0^1 \int_{\mathbb{R}^d} \nabla\varphi(x,t) \cdot dm(x,t)$$

We now seek v s.t. $dm = v_t d\mu_t dt$.

To see this, note that, since
$$dm^k = v^k d\mu_t^k dt \ll d\mu_t^k dt$$

$$\mathcal{B}(\mu_t^k dt, m^k) = \int_0^1 \int_{\mathbb{R}^d} |v^k|^2 d\mu_t^k dt.$$

Using lsc of $\mathcal{B}$,

$$\mathcal{B}(\mu_t dt, m) \leq \liminf_{k\to\infty} \mathcal{B}(\mu_t^k dt, m^k)$$

$\int_0^1 \int_{\mathbb{R}^d} |v_t|^2 d\mu_t dt$

$$= \liminf_{k\to\infty} \int_0^1 \int_{\mathbb{R}^d} |v^k|^2 d\mu_t^k dt.$$

$$\leq \int_0^1 |\mu'|^2(t) dt < +\infty$$

Therefore $m \ll d\mu_t dt$, so $\exists\, v$ s.t. $dm = v_t d\mu_t dt$.

Thus (μ, v) solves (CE).

It remains to show

$$\int |v_t|^2 d\mu_t \leq |\mu'|(t), \text{ a.e. } t \in [0,1]$$

It suffices to show that $\forall [a,b] \subseteq [0,1]$

$$\int_a^b \int_{\mathbb{R}^d} |v_t|^2 d\mu_t dt \leq \int_a^b |\mu'|(t)\, dt.$$

By lemma, the fact that

$$\left.\begin{array}{l} v_t^k \, d\mu_t^k \, dt \to v \, d\mu_t \, dt \\ d\mu_t^k \, dt \to d\mu_t \, dt \end{array}\right\} \text{narrowly}$$

Thus,

$$\left.\begin{array}{l} v_t^k \, d\mu_t^k \, dt\big|_{[a,b]} \to v \, d\mu_t \, dt\big|_{[a,b]} \\ d\mu_t^k \, dt\big|_{[a,b]} \to d\mu_t \, dt\big|_{[a,b]} \end{array}\right\} \text{narrowly}$$

Lower semicontinuity of $\mathcal{B}$ gives the result.

Next: part (ii).

Suppose (μ, v) solves (CE) and

$$\int_0^1 \int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x) \, dt < +\infty.$$

First, we will show $\mu \in AC^2(0,1; \mathcal{P}_2(\mathbb{R}^d))$.

Let $\varphi_\varepsilon : \mathbb{R}^d \to [0,+\infty)$ be a Gaussian with standard deviation $\varepsilon > 0$.

Let $\eta : \mathbb{R}^d \to [0,1]$ be a smooth, radially decreasing cutoff fn w/ $\eta \equiv 1$ on B_1, $\eta \equiv 0$ on B_2^c, $\|\nabla \eta\|_\infty \leq 1$.
Let $\eta_R(x) = \eta\left(\frac{x}{R}\right)$.