Lecture 9

Recall:

Def: Given nvs $\mathcal{X}$ and $\mathcal{U}$ and
perturbations
a proper and convex function $F: \mathcal{X} \times \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$,

primal problem: $P_0 := \inf_{x \in \mathcal{X}} f(x)$, $f(x) = F(x, 0)$

dual problem: $D_0 := \sup_{v \in \mathcal{U}^*} g(v)$, $g(v) = -F^*(0, v)$

Thm (Equivalence of Primal + Dual)
Given $F : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$ proper, convex, suppose $P_0 < +\infty$. ← "primal problem is feasible"

Define the inf-projection $P(u) = \inf_{x \in \mathcal{X}} F(x, u)$.

Then,
(i) $P_0 = D_0 \iff P$ is lsc at $u = 0$.
(ii) $P_0 = D_0$ and v^* is a maximizer of dual problem
$\iff v^* \in \partial P(0)$.

Kantorovich Duality

$(\mathcal{X}, d_{\mathcal{X}})$, (Y, d_Y) Polish spaces
$\mu \in \mathcal{P}(\mathcal{X})$, $\nu \in \mathcal{P}(Y)$
$c : \mathcal{X} \times Y \to \mathbb{R} \cup \{+\infty\}$ lsc, bdd below

$$\min_{\gamma : \gamma \in \Gamma(\mu,\nu)} \overbrace{\int_{X \times Y} c(x,y)\, d\gamma(x,y)}^{\mathbb{K}(\gamma)} \qquad (KP)$$

We can also write dual problem as a saddle point problem:

$$g(v) = -F^*(0,v) = -\sup_{(x,u) \in \mathcal{X} \times \mathcal{U}} \langle 0, x \rangle + \langle v, u \rangle - F(x,u)$$

$$= \inf_{(x,u) \in \mathcal{X} \times \mathcal{U}} F(x,u) - \langle v, u \rangle$$

$$\Rightarrow D_0 = \sup_{v \in \mathcal{U}^*} \inf_{(x,u) \in \mathcal{X} \times \mathcal{U}} F(x,u) - \langle v, u \rangle$$

Moral: Introducing a Lagrange multiplier to remove constraint can shed light on how to choose perturbation function $F(x,u)$.

<u>Fact</u>: If X is locally cpt, $\mu \in \mathcal{M}(X)$, then $C_c(X)$ are dense in $L^1(\mu)$. (See Folland Thm 7.8, 7.9.)

<u>Lemma</u>: Given $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, $\gamma \in \mathcal{M}(X \times Y)$,

$$\chi_{\Gamma(\mu,\nu)}(\gamma)$$

$$= \sup_{\varphi \in C_b(X),\, \psi \in C_b(Y)} \underbrace{\langle \mu - \pi^1 \# \gamma, \varphi \rangle + \langle \nu - \pi^2 \# \gamma, \psi \rangle}_{(*)}$$

Pf: The equality is clearly true if $\gamma \in \Gamma(\mu,\nu)$.

$\pi^1\#\gamma(A) = \gamma(A\times Y)$

If $\gamma \in \mathcal{M}(X\times Y)\setminus \mathcal{P}(X\times Y)$, then $1 \neq m_0 := \gamma(X\times Y) = \pi^1\#\gamma(X)$. Thus, $\mu \neq \pi^1\#\gamma$.

taking $\varphi_n(x) = n(1-m_0)$, $\psi(y) \equiv 0$,

$$(*) \geq \sup_n \langle \mu - \pi^1\#\gamma, \varphi_n \rangle + 0$$
$$= \sup_n \; n(1-m_0)\left(\int_X \mu - \int_X \pi^1\#\gamma\right)$$
$$= \sup_n \; n(1-m_0)^2$$
$$= +\infty$$

Hence, equality holds in Lemma statement.

Finally, if $\gamma \in \mathcal{P}(X \times Y) \setminus \Gamma(\mu, \nu)$, then either $\pi^1 \# \gamma \neq \mu$ or $\pi^2 \# \gamma \neq \nu$. WLOG $\pi^1 \# \gamma \neq \mu$.

Claim: there must exist $\varphi_0 \in C_c(X)$ s.t. $\underbrace{\int \varphi_0 \, d\pi^1 \# \gamma - \int \varphi_0 \, d\mu}_{c_0} \neq 0$.

Pf of Claim: Otherwise, by density of C_c in integrable fns, the measures would have to agree for all $A \in \mathcal{B}(X)$, hence be the same measure.

Let $\varphi_n(x) = -n c_0 \varphi_0(x)$, $\psi \equiv 0$. Then

$$
\begin{aligned}
(*) &\geq \sup_n \langle \mu - \pi^1 \# \gamma, \varphi_n \rangle + 0 \\
&= \sup_n (-n c_0)(-c_0) \\
&= +\infty .
\end{aligned}
$$

Proposition: Suppose $\mathcal{X}$ is a Polish space and $f: \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is lsc and bdd below. Then $\forall \mu \in \mathcal{P}(\mathcal{X})$,

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \in C_b(\mathcal{X}), \, g \leq f \right\}$$

Pf: Exercise 19

formerly $\mathcal{X}$ formerly x

How should we choose $Z, \mathcal{U}, F(z,u)$ so that KP coincides with...

$$D_0 = \sup_{v \in \mathcal{U}^*} \inf_{(z,u) \in Z \times \mathcal{U}} F(z,u) - \langle v, u \rangle \quad ?$$

In the case of KP...

$$-KP = -\inf_{\gamma\in\mathcal{M}(X\times Y)} \mathbb{K}_c(\gamma) + \chi_{\Gamma(\mu,\nu)}(\gamma)$$

$$= -\inf_{\gamma\in\mathcal{M}(X\times Y)} \sup_{(\varphi,\psi)\in C_b(X)\times C_b(Y)} \mathbb{K}_c(\gamma) + \langle \mu - \pi^1_\#\gamma, \varphi\rangle + \langle \nu - \pi^2_\#\gamma, \psi\rangle$$

$$= \sup_{\gamma\in\mathcal{M}(X\times Y)} \inf_{(\varphi,\psi)\in C_b(X)\times C_b(Y)} -\mathbb{K}_c(\gamma) + \langle \pi^1_\#\gamma - \mu, \varphi\rangle + \langle \pi^2_\#\gamma - \nu, \psi\rangle$$

Gathering the γ's...

$$-\mathbb{K}_c(\gamma) + \langle \pi^1_\#\gamma - \mu, \varphi\rangle + \langle \pi^2_\#\gamma - \nu, \psi\rangle$$

$$= -\int\varphi\, d\mu - \int\psi\, d\nu - \int \underbrace{c(x,y) - \varphi(\overbrace{\pi^1(x,y)}^{=x}) - \psi(\overbrace{\pi^2(x,y)}^{=y})}_{\text{lsc and bdd below}}\, d\gamma$$

$$= -\int\varphi\, d\mu - \int\psi\, d\nu - \sup_{\substack{u\in C_b(X\times Y),\\ u(x,y)\le c(x,y)-\varphi(x)-\psi(y)}} \int u(x,y)\, d\gamma$$

$$= \inf_{u \in C_b(X\times Y)} \underbrace{-\int \varphi \, d\mu - \int \psi \, d\nu + \chi_C(u)}_{F((\varphi,\psi),u)} - \int u(x,y)\, d\gamma$$

$C := \{u \in C_b(X\times Y) : u(x,y) \le c(x,y) - \varphi(x) - \psi(y)\}$

$\boxed{\text{THUS}}$

$-KP$

$$= \sup_{\gamma \in \mathcal{M}(X\times Y)} \inf_{(\varphi,\psi)\in C_b(X)\times C_b(Y)} \inf_{u \in C_b(X\times Y),} F((\varphi,\psi),u) - \langle u,\gamma\rangle$$

For simplicity, assume (X, d_X) and (Y, d_Y) are compact metric spaces.

Fact: $(C(X))^* = \mathcal{M}^s(X)$

What is the corresponding primal problem?

primal problem: $\mathcal{P}_0 := \inf_{x \in \mathcal{X}} f(x)$, $f(x) = F(x, 0)$

By defn of F,

$$\mathcal{P}_0 = \inf_{(\varphi,\psi) \in C_b(X) \times C_b(Y)} -\int \varphi \, d\mu - \int \psi \, d\nu + \chi_c(0)$$

$$= \inf_{\substack{(\varphi,\psi) \in C_b(X) \times C_b(Y) \\ 0 \le c(x,y) - \varphi(x) - \psi(y) \ \forall x \in X, y \in Y}} -\int \varphi \, d\mu - \int \psi \, d\nu$$

$$= -\sup_{\substack{(\varphi,\psi) \in C_b(X) \times C_b(Y) \\ \varphi(x) + \psi(y) \le c(x,y) \ \forall x \in X, y \in Y}} \int \varphi \, d\mu + \int \psi \, d\nu .$$

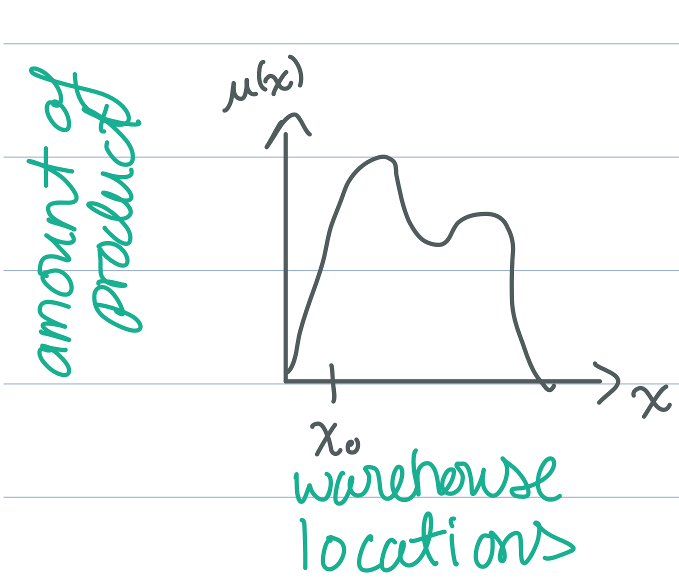
Optimal Transport MAD LIBS!

Name of retailer: Buckeys

Type of product: Silly string

Worthy cause: Espresso for grad offices

The Shipper's Problem (Caffarelli)



- It costs Buckeys $c(x, y)$ dollars to move one can of silly string from x to y.

- You want to make extra $$$ to support espresso for grad offices

- You charge Buckeys $\varphi(x)$ dollars to pick up one can of silly string from location x and $\psi(y)$ dollars to deliver to y.

Obviously, if Buckeys will let you ship, the following must be true:

$$\varphi(x) + \psi(y) \leq c(x,y).$$

- $$-P_0 = \sup_{\substack{(\varphi,\psi) \in C_b(X) \times C_b(Y) \\ \varphi(x)+\psi(y) \leq c(x,y)\ \forall x \in X,\, y \in Y.}} \int \varphi \, d\mu + \int \psi \, d\nu$$

$=$ largest amt of money you can make

- $-D_0 = \inf_{\gamma \in P(\mu,\nu)} \int c(x^1,x^2)\, d\gamma(x^1,x^2)$
  = least amt it would cost <u>Buckey's</u> to do it themselves

- We always have $P_0 \geq D_0 \Leftrightarrow -D_0 \geq -P_0$.
  If there is no duality gap, $P_0 = D_0$.

Now: prove $P_0 = D_0$ for (KP)

Thm: Suppose $(X, d_X), (Y, d_Y)$ are cpt Polish spaces and $c \in C(X \times Y)$, $c \geq 0$. Then $\forall$ $\mu \in P(X)$, $\nu \in P(Y)$,

$$\underbrace{\inf_{\gamma \in \Gamma(\mu,\nu)} \mathbb{K}(\gamma)}_{=-D_0} = \underbrace{\sup_{\substack{(\varphi,\psi)\in C(X)\times C(Y) \\ \varphi \oplus \psi \leq c}} \int \varphi \, d\mu + \int \psi \, d\nu}_{=-P_0}$$

Furthermore, the maximum is attained.

<u>Lemma</u>: Let (X, d_X) be a metric space and $\mathcal{A}$ a set. Suppose $\mathcal{F}$ is a collection of functions $f: X \times \mathcal{A} \to \mathbb{R}$. Suppose $\{f(\cdot,\alpha) : f \in \mathcal{F}, \alpha \in \mathcal{A}\}$ is e-cts. Then $\{\inf_{\alpha \in \mathcal{A}} f(\cdot,\alpha) : f \in \mathcal{F}\}$ is e-cts.

Exercise 19