Math CS 117: Homework 1

Due Sunday, April 7th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

DEFINITION 1 (composition)**.** Given $g : X \to Y$ and $f : Y \to Z$, the *composition* $f \circ g : X \to Z$ is the function defined by $f \circ g(x) = f(g(x))$ for all $x \in X$.

(a) Suppose g and f are one-to-one functions. Prove that $f \circ g$ is a one-to-one function and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

(b) Suppose g and f are onto functions. Prove that $f \circ g$ is an onto function.

(c) Suppose $A \subseteq Z$. Prove that $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A))$.

Question 2

Given $f : X \to Y$, prove that f is a one-to-one function iff $f(A \cap B) = f(A) \cap f(B)$ $\forall A, B \subseteq X$.

Question 3*

Suppose F is an ordered field and $x \in F$. Prove the following facts:

(a) $x \cdot 0 = 0$

(b) $-(-x) = x$

(c) $(-1)x = -x$

Question 4*

In the definition of a field, suppose that the condition $1 \neq 0$ in item M4 was removed. Prove that, if F is a field for which $1 = 0$, then $F = \{0\}$.

Question 5

Suppose F is an ordered field and $x, y, \epsilon \in F$.

(a) If $xy > 0$, prove that either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$.

(b) If $xy > 0$ and $x < y$, prove that $1/y < 1/x$.

(c) Prove that $x^2 > 0$ for all $x \neq 0$.

(d) Prove that $x^2 + y^2 \geq 2xy$.

(e) If $x \leq y + \epsilon$ for all $\epsilon > 0$, prove that $x \leq y$.

Question 6*

Consider the *Gaussian rational field* $\mathbb{Q}(i)$, defined to be the set

$$\mathbb{Q}(i) = \{p + qi : p, q \in \mathbb{Q}\},$$

where i denotes an element satisfying $i^2 = -1$. As the name implies, the Gaussian rational field is a field, endowed with the following addition and multiplication operations:

$$(p + qi) + (p' + q'i) = (p + p') + (q + q')i, \quad (p + qi) \cdot (p' + q'i) = (pp' - qq') + (pq' + qp')i.$$

The additive identity is $0 = 0 + 0i$ and the multiplicative identity is $1 = 1 + 0i$.

We may endow the Gaussian rational field with the *lexicographical ordering* given by

$$p + qi \leq p' + q'i \iff \text{ either (i) } p < p' \text{ or (ii) } p = p' \text{ and } q \leq q'.$$

(This is sometimes known as the *dictionary ordering*, since it follows the same principle by which one puts a list of words in alphabetical order.)

(a) For any $x, y \in \mathbb{Q}(i)$ prove that exactly one of the following is true: $x < y$, $x = y$ or $x > y$.

(b) For any $x, y, z \in \mathbb{Q}(i)$, if $x > y$ and $y > z$, prove that $x > z$.

(c) Even though $\mathbb{Q}(i)$ is a field and can be endowed with an ordering as described above, it is *not* an ordered field. Which part of the definition of an ordered field is violated? Justify your answer.