First, we consider part (i). There exist convex functions $f : \mathbb{R} \to \mathbb{R}$ so that $-f$ is not convex. For example, consider $f(x) = x^2$. We directly compute,

$$f((1-\alpha)x+\alpha y) = ((1-\alpha)x+\alpha y)^2 = (1-\alpha)^2 x^2 + \alpha^2 y^2 + 2(1-\alpha)\alpha xy = (1-\alpha)^2 f(x) + \alpha^2 f(y) + 2(1-\alpha)\alpha xy.$$  

Since $2xy \leq x^2 + y^2$ for all $x, y \in \mathbb{R}$, we have

$$f((1-\alpha)x + \alpha y) \leq (1 - \alpha)^2 f(x) + \alpha^2 f(y) + (1 - \alpha)\alpha (f(x) + f(y)) = (1 - \alpha)f(x) + \alpha f(y),$$

which shows $f$ is convex.

On the other hand, choosing $x = 0, y = 1, \alpha = \frac{1}{2}$, we have

$$-f((1 - \alpha)x + \alpha y) = -(\alpha y^2) = -\alpha^2 y^2 > 0 - \alpha y^2 = (1 - \alpha)f(x) + \alpha f(y).$$

Thus, $-f$ is not convex.

Now, we consider part (ii). We will show that $c_1 f + c_2 g$ is convex for all $f, g$ convex if and only if $c_1, c_2 \geq 0$. First, note that if $f(x)$ is convex, multiplying both sides of the inequality defining convexity by any positive constant $c > 0$ preserves the inequality,

$$f((1-\alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y) \implies cf((1 - \alpha)x + \alpha y) \leq (1 - \alpha)cf(x) + \alpha cf(y).$$

Next, note that, if $f$ and $g$ are convex, then summing the corresponding inequalities shows $f + g$ is convex: that is

$$f((1-\alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y) \quad \text{and} \quad f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y)$$

implies

$$(f + g)((1-\alpha)x + \alpha y) = f((1-\alpha)x + \alpha y) + g((1-\alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y) + (1 - \alpha)g(x) + \alpha g(y)$$

$$= (1 - \alpha)(f + g)(x) + \alpha (f + g)(y).$$

This shows that, if $c_1, c_2 \geq 0$, then $c_1 f(x) + c_2 g(x)$ is a convex function.

Now we argue that $c_1, c_2 \geq 0$ is necessary for $c_1 f(x) + c_2 g(x)$ to be convex for all $f, g$ convex. Note that the function $h(x) = 0$ is clearly convex. Thus, applying the result from part (i) shows that if $c_1 < 0$, then $c_1 f + c_2 g$ is not convex for $f(x) = -\frac{1}{c_1} x^2$, $g(x) = 0$, and if $c_2 < 0$, then $c_1 f + c_2 g$ is not convex for $f(x) = 0$ and $g(x) = -\frac{1}{c_1} x^2$.

We conclude by considering part (iii). First, we show that $g \circ f$ is convex. For all $x, y \in \mathbb{R}, \alpha \in [0, 1]$, the convexity of $f$ ensures,

$$f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y).$$

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Since $g$ is increasing, we have
\[ g((1 - \alpha)x + \alpha y) \leq g((1 - \alpha)f(x) + \alpha f(y)). \]

Since $g$ is convex, we may bound the right hand side from above by
\[ (1 - \alpha)g(f(x)) + \alpha g(f(y)). \]

This shows $g \circ f$ is convex.

Now, we show that $f \circ g$ is, in general, not convex. Let $f(x) = -x$ and $g(x) = \max\{x, 0\}$. A direct computation shows that $f$ is convex, since it is linear, so equality holds in the inequality defining convexity. Now we consider $g$. If $x \leq y$, then $\max\{x, 0\} \leq \max\{y, 0\}$, so $g$ is increasing. (This can be seen by considering the cases in which $x$ and $y$ are greater than or less than zero.) Finally, to see that $g$ is convex, we aim to show that, for all $x, y \in \mathbb{R}$, $\alpha \in [0,1]$,
\[ g((1 - \alpha)x + \alpha y) \leq (1 - \alpha)g(x) + \alpha g(y). \] (*)

If both $x$ and $y$ are less than zero, so is $(1 - \alpha)x + \alpha y$ so both sides of (*) are zero. Similarly, if both $x$ and $y$ are greater than or equal to zero, so is $(1 - \alpha)x + \alpha y$, so both sides of (*) equal $(1 - \alpha)x + \alpha y$. Finally, if $x < 0$ and $y \geq 0$, we consider the two cases when $(1 - \alpha)x + \alpha y$ is less than zero or greater than or equal to zero. If $(1 - \alpha)x + \alpha y$ is less than zero, then the left hand side of (*) is zero and the right hand side of (*) is $\alpha y \geq 0$. When $(1 - \alpha)x + \alpha y$ is greater than or equal to zero, the left hand side of (*) is $(1 - \alpha)x + \alpha y$ and the right hand side is
\[ \alpha y \geq (1 - \alpha)x + \alpha y, \]

since $(1 - \alpha) \geq 0$ and $x < 0$. 