Math 117: Homework 2

Q10 Solution

First, we consider part (i). There exist convex functions $f : \mathbb{R} \to \mathbb{R}$ so that $-f$ is not convex. For example, consider $f(x) = x^2$. We directly compute,

$$f((1-\alpha)x+\alpha y) = ((1-\alpha)x+\alpha y)^2 = (1-\alpha)^2x^2+\alpha^2y^2+2(1-\alpha)\alpha xy = (1-\alpha)^2f(x)+\alpha^2f(y)+2(1-\alpha)\alpha xy.$$

Since $2xy \leq x^2 + y^2$ for all $x, y \in \mathbb{R}$, we have

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)^2f(x) + \alpha^2f(y) + (1-\alpha)\alpha(f(x) + f(y) = (1-\alpha)f(x) + \alpha f(y),$$

which shows f is convex.
On the other hand, choosing $x = 0, y = 1$, $\alpha = \frac{1}{2}$, we have

$$-f((1-\alpha)x + \alpha y) = -(\alpha y^2) = -\alpha^2y^2 > 0 - \alpha y^2 = (1-\alpha)f(x) + \alpha f(y).$$

Thus, $-f$ is not convex.

Now, we consider part (ii). We will show that $c_1f + c_2g$ is convex for all f, g convex if and only if $c_1, c_2 \geq 0$. First, note that if $f(x)$ is convex, multiplying both sides of the inequality defining convexity by any positive constant $c > 0$ preserves the inequality,

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y) \implies cf((1-\alpha)x + \alpha y) \leq (1-\alpha)cf(x) + \alpha cf(y).$$

Next, note that, if f and g are convex, then summing the corresponding inequalities shows $f + g$ is convex: that is

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y) \quad \text{and} \quad f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y)$$

implies

$$\begin{aligned}(f+g)((1-\alpha)x + \alpha y) &= f((1-\alpha)x + \alpha y) + g((1-\alpha)x + \alpha y) \\ &\leq (1-\alpha)f(x) + \alpha f(y) + (1-\alpha)g(x) + \alpha g(y) \\ &= (1-\alpha)(f+g)(x) + \alpha(f+g)(y).\end{aligned}$$

This shows that, if $c_1, c_2 \geq 0$, then $c_1f(x) + c_2g(x)$ is a convex function.

Now we argue that $c_1, c_2 \geq 0$ is necessary for $c_1f(x) + c_2g(x)$ to be convex for all f, g convex. Note that the function $h(x) = 0$ is clearly convex. Thus, applying the result from part (i) shows that if $c_1 < 0$, then $c_1f + c_2g$ is not convex for $f(x) = -\frac{1}{c_1}x^2$, $g(x) = 0$, and if $c_2 < 0$, then $c_1f + c_2g$ is not convex for $f(x) = 0$ and $g(x) = -\frac{1}{c_1}x^2$.

We conclude by considering part (iii). First, we show that $g \circ f$ is convex. For all $x, y \in \mathbb{R}$, $\alpha \in [0, 1]$, the convexity of f ensures,

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y).$$

Since g is increasing, we have

$$g\left(f((1-\alpha)x+\alpha y)\right) \leq g\left((1-\alpha)f(x)+\alpha f(y)\right).$$

Since g is convex, we may bound the right hand side from above by

$$(1-\alpha)g(f(x))+\alpha g(f(y)).$$

This shows $g \circ f$ is convex.

Now, we show that $f \circ g$ is, in general, not convex. Let $f(x) = -x$ and $g(x) = \max\{x, 0\}$. A direct computation shows that f is convex, since it is linear, so equality holds in the inequality defining convexity. Now we consider g. If $x \leq y$, then $\max\{x, 0\} \leq \max\{y, 0\}$, so g is increasing. (This can be seen by considering the cases in which x and y are greater than or less than zero.) Finally, to see that g is convex, we aim to show that, for all $x, y \in \mathbb{R}$, $\alpha \in [0, 1]$,

$$g((1-\alpha)x+\alpha y) \leq (1-\alpha)g(x)+\alpha g(y). \tag{*}$$

If both x and y are less than zero, so is $(1-\alpha)x+\alpha y$ so both sides of (*) are zero. Similarly, if both x and y are greater than or equal to zero, so is $(1-\alpha)x+\alpha y$, so both sides of (*) equal $(1-\alpha)x+\alpha y$. Finally, if $x < 0$ and $y \geq 0$, we consider the two cases when $(1-\alpha)x+\alpha y$ is less than zero or greater than or equal to zero. If $(1-\alpha)x+\alpha y$ is less than zero, then the left hand side of (*) is zero and the right hand side of (*) is $\alpha y \geq 0$. When $(1-\alpha)x+\alpha y$ is greater than or equal to zero, the left hand side of (*) is $(1-\alpha)x+\alpha y$ and the right hand side is

$$\alpha y \geq (1-\alpha)x+\alpha y,$$

since $(1-\alpha) \geq 0$ and $x < 0$.

