Math 117: Homework 4

Due Sunday, April 28th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Which of the following is an equivalent definition of convergence of a sequence?

A sequence s_n converges to a limit s if...

(a) there exists $\epsilon > 0$ so that, for all $N \in \mathbb{R}$, $n > N$ ensures $|s_n - s| < \epsilon$.

(b) for all $\epsilon \geq 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \epsilon$.

(c) for all $\epsilon > 0$, $|s_n - s| < \epsilon$ for all $n \in \mathbb{N}$.

(d) given $\epsilon > 0$, there is some $N \in \mathbb{R}$ so that $|s_n - s| < \epsilon$ for all $n > N$.

For each of the statements above that are *not* the correct definition of convergence, give an example of either

- a sequence that satisfies the statement but does not converge or
- a sequence that converges but does not satisfy the statement.

In each case, the existence of such an example will illustrate why the statement is *not* equivalent to the correct definition of convergence.

Question 2

Solve exercise 12.2 in the textbook.

Question 3*

Solve exercise 12.4 in the textbook.

Question 4*

In this problem, you will prove that, for any $a \in \mathbb{R}$, $a \geq 0$, there exists $b \in \mathbb{R}$, $b \geq 0$ so that $b^2 = a$. This number b is known as *the square root* of a and is denoted $\sqrt{a}$. Note that, if $a = 0$, we can directly see that $0 \cdot 0 = 0$, so $\sqrt{a} = 0$. For the rest of the problem, suppose $a > 0$.

(i) Suppose $x, y \geq 0$ with $x^2 < a$ and $y^2 > a$. Prove that there exists $\epsilon_1, \epsilon_2 \in (0, 1)$ so that $(x + \epsilon_1)^2 < a$ and $(y - \epsilon_2)^2 > a$.

(ii) Prove that the set $S = \{z \in \mathbb{R} : z \geq 0, z^2 \leq a\}$ is nonempty and bounded above, so that its supremum exists.

(iii) Prove that $b^2 = a$. (Hint: use the definition of supremum to prove $b^2 \geq a$ and $b^2 \leq a$.)

Question 5*

Solve exercise 12.6 in the textbook.

Question 6

Solve exercise 12.7 in the textbook.

Question 7

Solve exercise 13.2 in the textbook.

Question 8*

Solve exercise 13.3 in the textbook. Make sure you justify that your examples by proving they are convergent/bounded/divergent.

Question 9

Solve exercise 13.4 in the textbook.

Question 10*

(a) Suppose a_n and b_n are convergent sequences and $a_n \leq b_n$ for all but finitely many $n \in \mathbb{N}$. Prove that $\lim_{n\to+\infty} a_n \leq \lim_{n\to+\infty} b_n$.

(b) Does the result of part a continue to hold if we replace both $\leq$ with $<$? Justify your answer with a proof or counterexample.

(c) Suppose $c, d \in \mathbb{R}$ and $c \leq d + \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that $c \leq d$. (Compare this to HW3, Q4.)

Question 11*

An important lemma in the analysis of sequences is known as the *squeeze lemma.*

LEMMA 1 (Squeeze Lemma)**.** *Consider three sequences a_n, b_n, and s_n. If $a_n \leq s_n \leq b_n$ for all but finitely many $n \in \mathbb{N}$ and there exists $s \in \mathbb{R}$ so that*

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n = s,$$

then $\lim_{n\to+\infty} s_n = s$.

In this question, you will prove the squeeze lemma and consider an important consequene of this lemma.

(a) Prove the squeeze lemma.

(b) Suppose s_n and t_n are sequences such that $|s_n| \leq t_n$ for all $n \in \mathbb{N}$. Prove that, if $\lim_{n\to+\infty} t_n = 0$, then $\lim_{n\to+\infty} s_n = 0$.

(c) Is the converse to part (b) true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 12

We now recall the definition of an *affine* set.

DEFINITION 1. A set $C \subseteq \mathbb{R}^d$ is *affine* if the line through any two points in C lies in C, that is, for all $x_0, x_1 \in C$,

$$(1-\alpha)x_0 + \alpha x_1 \in C, \quad \forall \alpha \in \mathbb{R}.$$

(a) Give an example of $C \subseteq \mathbb{R}^d$ that is affine but is not a subspace.

(b) Given $\{x_1, \ldots, x_k\} \subseteq \mathbb{R}^d$, any point of the form

$$\sum_{i=1}^{k} \alpha_i x_i, \text{ such that } \sum_{i=1}^{k} \alpha_i = 1,$$

is an *affine combination* of $\{x_i\}_{i=1}^k$. If C is affine, prove that C is *closed under affine combinations*, that is, for any $k \in \mathbb{N}$, $\{x_i\}_{i=1}^k \subseteq C$, all affine combinations of $\{x_i\}_{i=1}^k$ are in C.

(c) If C is an affine set and $x_0 \in C$, prove that

$$C - x_0 := \{x - x_0 : x \in C\}$$

is a subspace.

Question 13*

We now recall the definition of a *convex* set.

DEFINITION 2. A set $C \subseteq \mathbb{R}^d$ is *convex* if the line segment between any two points in C lies in C, that is, for all $x_0, x_1 \in C$,

$$(1-\alpha)x_0 + \alpha x_1 \in C, \quad \forall \alpha \in [0,1].$$

(a) Give an example of $C \subseteq \mathbb{R}^d$ that is convex but not affine.

(b) Given $\{x_1, \ldots, x_k\} \subseteq \mathbb{R}^d$, any point of the form

$$\sum_{i=1}^{k} \alpha_i x_k, \text{ such that } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{k} \alpha_i = 1,$$

is a *convex combination* of $\{x_i\}_{i=1}^k$. If C is convex, prove that C is *closed under convex combinations*, that is, for any $k \in \mathbb{N}$, $\{x_i\}_{i=1}^k \subseteq C$, all convex combinations of $\{x_i\}_{i=1}^k$ are in C. (A convex combination can be thought of as a weighted average of the points.)

