Math 117: Homework 4
Due Sunday, April 28th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Which of the following is an equivalent definition of convergence of a sequence?

A sequence \( s_n \) converges to a limit \( s \) if...

(a) there exists \( \epsilon > 0 \) so that, for all \( N \in \mathbb{R} \), \( n > N \) ensures \( |s_n - s| < \epsilon \).

(b) for all \( \epsilon \geq 0 \), there exists \( N \in \mathbb{R} \) so that \( n > N \) ensures \( |s_n - s| < \epsilon \).

(c) for all \( \epsilon > 0 \), \( |s_n - s| < \epsilon \) for all \( n \in \mathbb{N} \).

(d) given \( \epsilon > 0 \), there is some \( N \in \mathbb{R} \) so that \( |s_n - s| < \epsilon \) for all \( n > N \).

For each of the statements above that are not the correct definition of convergence, give an example of either

- a sequence that satisfies the statement but does not converge or
- a sequence that converges but does not satisfy the statement.

In each case, the existence of such an example will illustrate why the statement is not equivalent to the correct definition of convergence.

Question 2

Solve exercise 12.2 in the textbook.

Question 3*

Solve exercise 12.4 in the textbook.

Question 4*

In this problem, you will prove that, for any \( a \in \mathbb{R}, a \geq 0 \), there exists \( b \in \mathbb{R}, b \geq 0 \) so that \( b^2 = a \). This number \( b \) is known as the square root of \( a \) and is denoted \( \sqrt{a} \). Note that, if \( a = 0 \), we can directly see that \( 0 \cdot 0 = 0 \), so \( \sqrt{a} = 0 \). For the rest of the problem, suppose \( a > 0 \).

(i) Suppose \( x, y \geq 0 \) with \( x^2 < a \) and \( y^2 > a \). Prove that there exists \( \epsilon_1, \epsilon_2 \in (0, 1) \) so that \( (x + \epsilon_1)^2 < a \) and \( (y - \epsilon_2)^2 > a \).

(ii) Prove that the set \( S = \{z \in \mathbb{R} : z \geq 0, z^2 \leq a \} \) is nonempty and bounded above, so that its supremum exists.

(iii) Prove that \( b^2 = a \). (Hint: use the definition of supremum to prove \( b^2 \geq a \) and \( b^2 \leq a \).)
Question 5*  
Solve exercise 12.6 in the textbook.

Question 6  
Solve exercise 12.7 in the textbook.

Question 7  
Solve exercise 13.2 in the textbook.

Question 8*  
Solve exercise 13.3 in the textbook. Make sure you justify that your examples by proving they are convergent/bounded/divergent.

Question 9  
Solve exercise 13.4 in the textbook.

Question 10*  
(a) Suppose \( a_n \) and \( b_n \) are convergent sequences and \( a_n \leq b_n \) for all but finitely many \( n \in \mathbb{N} \). Prove that \( \lim_{n \to +\infty} a_n \leq \lim_{n \to +\infty} b_n \).

(b) Does the result of part a continue to hold if we replace both \( \leq \) with \( < \)? Justify your answer with a proof or counterexample.

(c) Suppose \( c, d \in \mathbb{R} \) and \( c \leq d + \frac{1}{n} \) for all \( n \in \mathbb{N} \). Prove that \( c \leq d \). (Compare this to HW3, Q4.)

Question 11*  
An important lemma in the analysis of sequences is known as the squeeze lemma.

**Lemma 1** (Squeeze Lemma). Consider three sequences \( a_n, b_n, \) and \( s_n \). If \( a_n \leq s_n \leq b_n \) for all but finitely many \( n \in \mathbb{N} \) and there exists \( s \in \mathbb{R} \) so that

\[
\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = s,
\]

then \( \lim_{n \to +\infty} s_n = s \).

In this question, you will prove the squeeze lemma and consider an important consequence of this lemma.

(a) Prove the squeeze lemma.

(b) Suppose \( s_n \) and \( t_n \) are sequences such that \( |s_n| \leq t_n \) for all \( n \in \mathbb{N} \). Prove that, if \( \lim_{n \to +\infty} t_n = 0 \), then \( \lim_{n \to +\infty} s_n = 0 \).

(c) Is the converse to part (b) true? If so, prove it. If not, give a counterexample and justify your counterexample.
Question 12

We now recall the definition of an **affine set**.

**DEFINITION 1.** A set $C \subseteq \mathbb{R}^d$ is affine if the line through any two points in $C$ lies in $C$, that is, for all $x_0, x_1 \in C$,

$$(1 - \alpha)x_0 + \alpha x_1 \in C, \quad \forall \alpha \in \mathbb{R}.$$ 

(a) Give an example of $C \subseteq \mathbb{R}^d$ that is affine but is not a subspace.

(b) Given $\{x_1, \ldots, x_k\} \subseteq \mathbb{R}^d$, any point of the form

$$\sum_{i=1}^{k} \alpha_i x_i, \text{ such that } \sum_{i=1}^{k} \alpha_i = 1,$$ 

is an **affine combination** of $\{x_i\}_{i=1}^{k}$. If $C$ is affine, prove that $C$ is closed under affine combinations, that is, for any $k \in \mathbb{N}$, $\{x_i\}_{i=1}^{k} \subseteq C$, all affine combinations of $\{x_i\}_{i=1}^{k}$ are in $C$.

(c) If $C$ is an affine set and $x_0 \in C$, prove that

$$C - x_0 := \{x - x_0 : x \in C\}$$

is a subspace.

Question 13*

We now recall the definition of a **convex set**.

**DEFINITION 2.** A set $C \subseteq \mathbb{R}^d$ is convex if the line segment between any two points in $C$ lies in $C$, that is, for all $x_0, x_1 \in C$,

$$(1 - \alpha)x_0 + \alpha x_1 \in C, \quad \forall \alpha \in [0, 1].$$

(a) Give an example of $C \subseteq \mathbb{R}^d$ that is convex but not affine.

(b) Given $\{x_1, \ldots, x_k\} \subseteq \mathbb{R}^d$, any point of the form

$$\sum_{i=1}^{k} \alpha_i x_k, \text{ such that } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{k} \alpha_i = 1,$$ 

is a **convex combination** of $\{x_i\}_{i=1}^{k}$. If $C$ is convex, prove that $C$ is closed under convex combinations, that is, for any $k \in \mathbb{N}$, $\{x_i\}_{i=1}^{k} \subseteq C$, all convex combinations of $\{x_i\}_{i=1}^{k}$ are in $C$.

(A convex combination can be thought of as a weighted average of the points.)