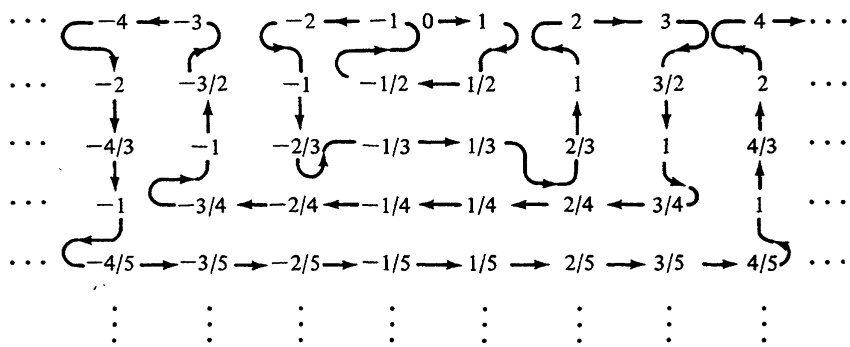
Math 117: Homework 6

Due Sunday, May 19th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

One can show that the set of rational numbers $\mathbb{Q}$ can be listed as a sequence r_n. The exact procedure is a little tedious, but you can get an idea of how it works by considering the below diagram.



For example, $r_1 = 0, r_2 = 1, r_3 = 1/2$, and so on. Note that some numbers, such as -1, are included multiples times.

(a) For any $\epsilon > 0$ and $a \in \mathbb{R}$, show that the set $\{r \in \mathbb{Q} : |r - a| < \epsilon\}$ contains infinitely many elements. (**Hint**: Use denseness of the rationals.)

(b) Let r_n be the sequence of rational numbers. Use part (a) to show that for any $a \in \mathbb{R}$, there exists a subsequence r_{n_k} that converges to a. (**Hint**: Use part (a) to show that the set $\{n \in \mathbb{N} : |r_n - a| < \epsilon\}$ is infinite.)

(c) Let r_n be the sequence of rational numbers. Show that there exists a subsequence r_{n_k} satisfying $\lim_{k\to+\infty} r_{n_k} = +\infty$.

Question 2* (decimal expansions)

In this problem you will show that any number that can be represented as a nonnegative decimal expansion can be thought of as the limit of a bounded increasing sequence of real numbers. Since all bounded monotone sequences converge, this guarantees that any decimal expansion you can imagine represents (converges to) a real number.

Suppose we are given a decimal expansion $K.d_1d_2d_3d_4\ldots$, where K is a nonnegative integer and each $d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let

$$s_n = K + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.$$

(a) Show s_n is an increasing sequence. (This is almost obvious. Your proof should be short.)

(b) Use a result from the previous homework to prove that $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$.

(c) Use part (b) to prove that s_n is a bounded sequence.

(d) Since $0.\bar{9} = 0.999\ldots$ and 1 are both decimal expansions, by what you have shown, they both correspond to a real number. Use the hint from part (b) to show that they actually correspond to the same real number.

Question 3

Solve 18.1 from the textbook.

Question 4

Solve 19.4 from the textbook

Question 5*

Consider two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ with

$$0 \leq a_k \leq b_k \text{ for all } k \in \mathbb{N}.$$

If $\sum_{k=1}^{\infty} a_k = +\infty$, prove that $\sum_{k=1}^{\infty} b_k = +\infty$.

Question 6

Suppose $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ for $A, B \in \mathbb{R}$.

(a) Use the limit theorems for sequences to prove that $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

(b) Use the limit theorems for sequences to prove that for $c \in \mathbb{R}$, $\sum_{k=1}^{\infty} ca_k = cA$.

Question 7* (absolute value of a series)

In general, the expression $\sum_{k=1}^{\infty} a_k$ doesn't always have meaning, since the limit of the corresponding sequence $s_n = \sum_{k=1}^{n} a_k$ doesn't always exist. On the other hand, in this problem you will show that the expression $\sum_{k=1}^{\infty} |a_k|$ always has meaning.

(a) Prove that $\sum_{k=1}^{\infty} |a_k|$ is either convergent or diverges to $+\infty$.

(Hint: Show that the corresponding sequence $s_n = \sum_{k=1}^{n} |a_k|$ is monotone.)

(b) Prove that if $\sum_{k=1}^{\infty} |a_k|$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is also convergent.

(Hint: First explain why $|\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k|$. Combine this fact with the Cauchy Criterion.)

Question 8

Solve 30.3 from the textbook.

Question 9

Solve 30.5 from the textbook.

Question 10*

Solve 30.6 from the textbook.

Question 11

Solve 30.8 from the textbook.

Question 12*

Given a function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, its *epigraph* is defined to be the set

$$\text{epi}(f) := \{(x, t) \in \mathbb{R} \times \mathbb{R} : t \geq f(x)\}.$$

Prove that f is a convex function if and only if $\text{epi}(f)$ is a convex set.