Math 117: Homework 6
Due Sunday, May 19th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

One can show that the set of rational numbers \( \mathbb{Q} \) can be listed as a sequence \( r_n \). The exact procedure is a little tedious, but you can get an idea of how it works by considering the below diagram.

For example, \( r_1 = 0, r_2 = 1, r_3 = 1/2, \) and so on. Note that some numbers, such as \(-1\), are included multiples times.

(a) For any \( \epsilon > 0 \) and \( a \in \mathbb{R} \), show that the set \( \{ r \in \mathbb{Q} : |r - a| < \epsilon \} \) contains infinitely many elements. (Hint: Use denseness of the rationals.)

(b) Let \( r_n \) be the sequence of rational numbers. Use part (a) to show that for any \( a \in \mathbb{R} \), there exists a subsequence \( r_{n_k} \) that converges to \( a \). (Hint: Use part (a) to show that the set \( \{ n \in \mathbb{N} : |r_n - a| < \epsilon \} \) is infinite.)

(c) Let \( r_n \) be the sequence of rational numbers. Show that there exists a subsequence \( r_{n_k} \) satisfying \( \lim_{k \to +\infty} r_{n_k} = +\infty \).

Question 2* (decimal expansions)

In this problem you will show that any number that can be represented as a nonnegative decimal expansion can be thought of as the limit of a bounded increasing sequence of real numbers. Since all bounded monotone sequences converge, this guarantees that any decimal expansion you can imagine represents (converges to) a real number.

Suppose we are given a decimal expansion \( K.d_1d_2d_3d_4 \ldots \), where \( K \) is a nonnegative integer and each \( d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Let

\[
s_n = K + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.
\]
(a) Show \( s_n \) is an increasing sequence. (This is almost obvious. Your proof should be short.)

(b) Use a result from the previous homework to prove that \( \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n} \).

(c) Use part (b) to prove that \( s_n \) is a bounded sequence.

(d) Since 0.9 = 0.999… and 1 are both decimal expansions, by what you have shown, they both correspond to a real number. Use the hint from part (b) to show that they actually correspond to the same real number.

**Question 3**

Solve 18.1 from the textbook.

**Question 4**

Solve 19.4 from the textbook

**Question 5**

Consider two series \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) with

\[
0 \leq a_k \leq b_k \quad \text{for all } k \in \mathbb{N}.
\]

If \( \sum_{k=1}^{\infty} a_k = +\infty \), prove that \( \sum_{k=1}^{\infty} b_k = +\infty \).

**Question 6**

Suppose \( \sum_{k=1}^{\infty} a_k = A \) and \( \sum_{k=1}^{\infty} b_k = B \) for \( A, B \in \mathbb{R} \).

(a) Use the limit theorems for sequences to prove that \( \sum_{k=1}^{\infty} (a_k + b_k) = A + B \).

(b) Use the limit theorems for sequences to prove that for \( c \in \mathbb{R}, \sum_{k=1}^{\infty} ca_k = cA \).

**Question 7**

In general, the expression \( \sum_{k=1}^{\infty} a_k \) doesn’t always have meaning, since the limit of the corresponding sequence \( s_n = \sum_{k=1}^{n} a_k \) doesn’t always exist. On the other hand, in this problem you will show that the expression \( \sum_{k=1}^{\infty} |a_k| \) always has meaning.

(a) Prove that \( \sum_{k=1}^{\infty} |a_k| \) is either convergent or diverges to \(+\infty\).

(Hint: Show that the corresponding sequence \( s_n = \sum_{k=1}^{n} |a_k| \) is monotone.)

(b) Prove that if \( \sum_{k=1}^{\infty} |a_k| \) is convergent, then \( \sum_{k=1}^{\infty} a_k \) is also convergent.

(Hint: First explain why \( |\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k| \). Combine this fact with the Cauchy Criterion.)
Question 8
Solve 30.3 from the textbook.

Question 9
Solve 30.5 from the textbook.

Question 10*
Solve 30.6 from the textbook.

Question 11
Solve 30.8 from the textbook.

Question 12*
Given a function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, its epigraph is defined to be the set
\[ \text{epi}(f) := \{(x, t) \in \mathbb{R} \times \mathbb{R} : t \geq f(x)\}. \]

Prove that $f$ is a convex function if and only if $\text{epi}(f)$ is a convex set.