Math 117: Homework 7

Due Sunday, June 2nd at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Prove the following theorem.

THEOREM 1. *Consider* $x_n : \mathbb{N} \to \mathbb{R}$.

(i) Fix $x \in \mathbb{R}$. *Then* x *is a subsequential limit of* x_n *if and only if the set* $\{n : |x_n - x| < \epsilon\}$ *is infinite for all* $\epsilon > 0$.

(ii) $+\infty$ *is a subsequential limit of* x_n *if and only if* x_n *is unbounded above.*

(iii) $-\infty$ *is a subsequential limit of* x_n *if and only if* x_n *is unbounded below.*

This has been corrected in the Lecture 15 and the version of this for the extended real numbers has been written in the Lecture 16 notes

Question 2

In lecture 12, we stated the following theorem.

THEOREM 2. *If* a *is a right and left accumulation point of* $X \subseteq \mathbb{R}$, *then*

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

We proved the implication $\Longrightarrow$. *Then we proved the implication* $\Longleftarrow$ *in the special case* $L \in \mathbb{R}$. *In this problem, you will show the result continues to hold even if* $L \in \{-\infty, +\infty\}$.

(i) Prove that $x_n : \mathbb{N} \to \overline{\mathbb{R}}$ diverges to $+\infty$ if and only if every subsequence x_{n_k} has a further subsequence $x_{n_{k_l}}$ that diverges to $+\infty$.

(ii) Use the fact from part (i) to adapt the proof from class to the case $L \in \{-\infty, +\infty\}$.

Question 3*

Consider the following definition of right continuity.

DEFINITION 1. A function $f : X \to \mathbb{R}$ is *right continuous* at $a \in X$ if either

(i) a is not a right accumulation point of X or

(ii) a is a right accumulation point of X and $\lim_{x \to a^+} f(x) = f(a)$.

(a) Prove that a is an accumulation point of a set $X \subseteq \mathbb{R}$ if and only if either a is a left accumulation point of X or a is a right accumulation point of X.

(b) Use the above definition to define what it means for a function to be left-continuous at a point in its domain.

(c) Prove that $f : X \to \mathbb{R}$ is continuous at $a \in X$ if and only if it is right and left continuous at a.

(d) Give an example of a function $f : [0,1] \to \mathbb{R}$ that is right continuous at x for all $x \in [0,1]$, but for which there exists $x_0 \in [0,1]$ so that f is not lower semicontinuous at x_0. Justify your answer.