Math 117: Homework 8

Due Sunday, June 9th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Suppose f is continuous on $\mathbb{R}$ and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists $a \in \mathbb{R}$ so that $f(x) = ax$.

Question 2*

Solve J&P 34.4.

Question 3*

Determine whether the following statements are true or false. If they are true, prove them. If they are false, give a counterexample and justify it.

(a) If a sequence s_n satisfies $\limsup s_n = 2$, then $s_n > 1.99$ for all n large enough.

(b) If a sequence s_n satisfies $\limsup s_n = b$, then $s_n \leq b$ for all n large enough.

Question 4

Suppose s_n and t_n are bounded sequences.

(a) Prove that $\limsup s_n + t_n \leq \limsup s_n + \limsup t_n$.

(b) Give an examples of bounded sequences s_n and t_n for which $\limsup s_n + t_n < \limsup s_n + \limsup t_n$.

Question 5

Suppose $f : \mathbb{R} \to \mathbb{R}$ is both lower semicontinuous and upper semicontinuous at x_0. Prove that f is continuous at x_0.

Question 6*

Suppose $f : [a, b] \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous at x for all $x \in [a, b]$.

(a) Prove that f is bounded below on $[a, b]$.

(b) Prove that there exists $x_* \in [a, b]$ so that $f(x_*) = \inf\{f(x) : x \in [a, b]\}$.

(c) Now suppose, in addition, that f is *strictly convex*, that is,

$$f((1-\alpha)x + \alpha y) < (1-\alpha)f(x) + \alpha f(y), \quad \forall \alpha \in (0, 1).$$

Prove that the x_* found in part (b) is unique.