Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

**Question 1**

Suppose $f$ is continuous on $\mathbb{R}$ and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists $a \in \mathbb{R}$ so that $f(x) = ax$.

**Question 2***

Solve J&P 34.4.

**Question 3***

Determine whether the following statements are true or false. If they are true, prove them. If they are false, give a counterexample and justify it.

(a) If a sequence $s_n$ satisfies $\limsup s_n = 2$, then $s_n > 1.99$ for all $n$ large enough.

(b) If a sequence $s_n$ satisfies $\limsup s_n = b$, then $s_n \leq b$ for all $n$ large enough.

**Question 4**

Suppose $s_n$ and $t_n$ are bounded sequences.

(a) Prove that $\limsup s_n + t_n \leq \limsup s_n + \limsup t_n$.

(b) Give an examples of bounded sequences $s_n$ and $t_n$ for which $\limsup s_n + t_n < \limsup s_n + \limsup t_n$.

**Question 5**

Suppose $f : \mathbb{R} \to \mathbb{R}$ is both lower semicontinuous and upper semicontinuous at $x_0$. Prove that $f$ is continuous at $x_0$.

**Question 6***

Suppose $f : [a, b] \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous at $x$ for all $x \in [a, b]$.

(a) Prove that $f$ is bounded below on $[a, b]$.

(b) Prove that there exists $x_* \in [a, b]$ so that $f(x_*) = \inf\{f(x) : x \in [a, b]\}$.

(c) Now suppose, in addition, that $f$ is strictly convex, that is,

$$f((1 - \alpha)x + \alpha y) < (1 - \alpha)f(x) + \alpha f(y), \quad \forall \alpha \in (0, 1).$$

Prove that the $x_*$ found in part (b) is unique.