Lecture 11

Office Hours:

Wed 3:30-4:30pm, Thur 1-2pm

This Friday, last makeup lecture 3:30-4:45pm

30 Definition of the Limit of a Function

Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, a is an accumulation point of X if $\forall \delta > 0$, $\exists x \in X$ s.t.

$$0 < |x - a| < \delta$$

Lemma: a is an accumulation point of $X \subseteq \mathbb{R}$ $\iff$ $\exists x_n : \mathbb{N} \to \mathbb{R}$ s.t. $x_n \in X \setminus \{a\}$ $\forall n \in \mathbb{N}$ and $x_n \to a$.

Def: Given $X \subseteq \mathbb{R}$ nonempty, $f: X \to \overline{\mathbb{R}}$, a an accumulation point of X, $L \in \overline{\mathbb{R}}$, the limit of $f(x)$ as x approaches a is L if, for all sequences $x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \to a$, we have

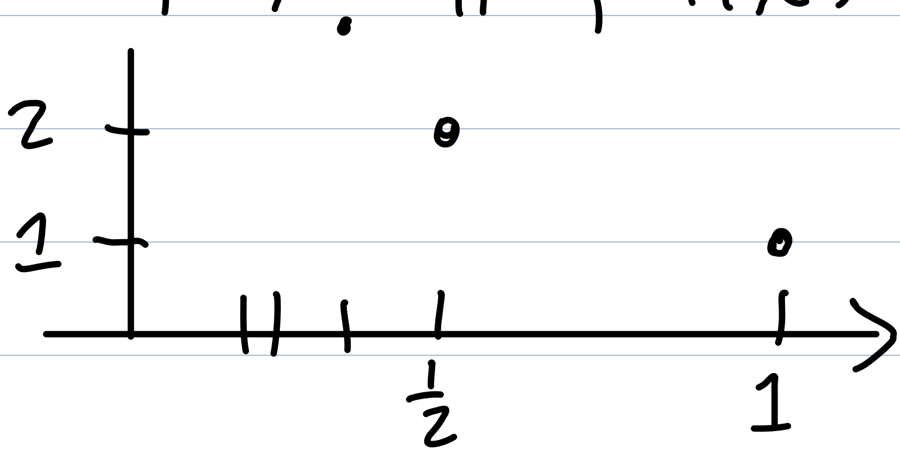
$$\lim_{n\to\infty} f(x_n) = L.$$

← "sequential def'n of $\lim_{x\to a} f(x)$"

We denote this as $\lim_{x\to a} f(x) = L$.

Ex: $X = \{\frac{1}{n} : n \in \mathbb{N}\}$

$f: X \to \overline{\mathbb{R}}$, $f(x) = \frac{1}{x}$



$a = 0$ is acc pt of X

$\lim_{x\to 0} f(x) = +\infty$

Let's prove this!

Fix $x_n : \mathbb{N} \to X$, s.t. $x_n \to 0$.

We must show $\lim_{n\to\infty} f(x_n) = +\infty$.

Fix $M \in \mathbb{R}$.

$$\boxed{\begin{array}{l}\text{Scratch: WLOG } M > 0. \\ \frac{1}{x_n} = f(x_n) > M \iff \frac{1}{M} > x_n\end{array}}$$

If $M \leq 0$, then $f(x_n) \geq M$ $\forall n \in \mathbb{N}$ and we are done. Suppose $M > 0$. Since $x_n \to 0$, $\exists$ N s.t. $n \geq N$ ensures $x_n = |x_n - 0| < \frac{1}{M}$.

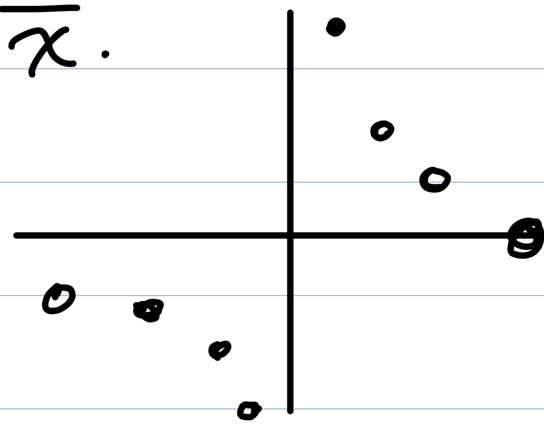
Thus $n \geq N$, $f(x_n) = \frac{1}{x_n} > M$.

Hence $\lim_{n\to\infty} f(x_n) = +\infty$.

$\{x : X = \{\frac{1}{m} : m \in \mathbb{Z}, m \neq 0\}\}$

$f : X \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

$\lim_{x \to 0} f(x)$ D.N.E.



<u>Prop</u>: Given $X \subseteq \mathbb{R}$ nonempty, $f : X \to \mathbb{R}$, a an acc point of X, and $L \in \mathbb{R}$, then

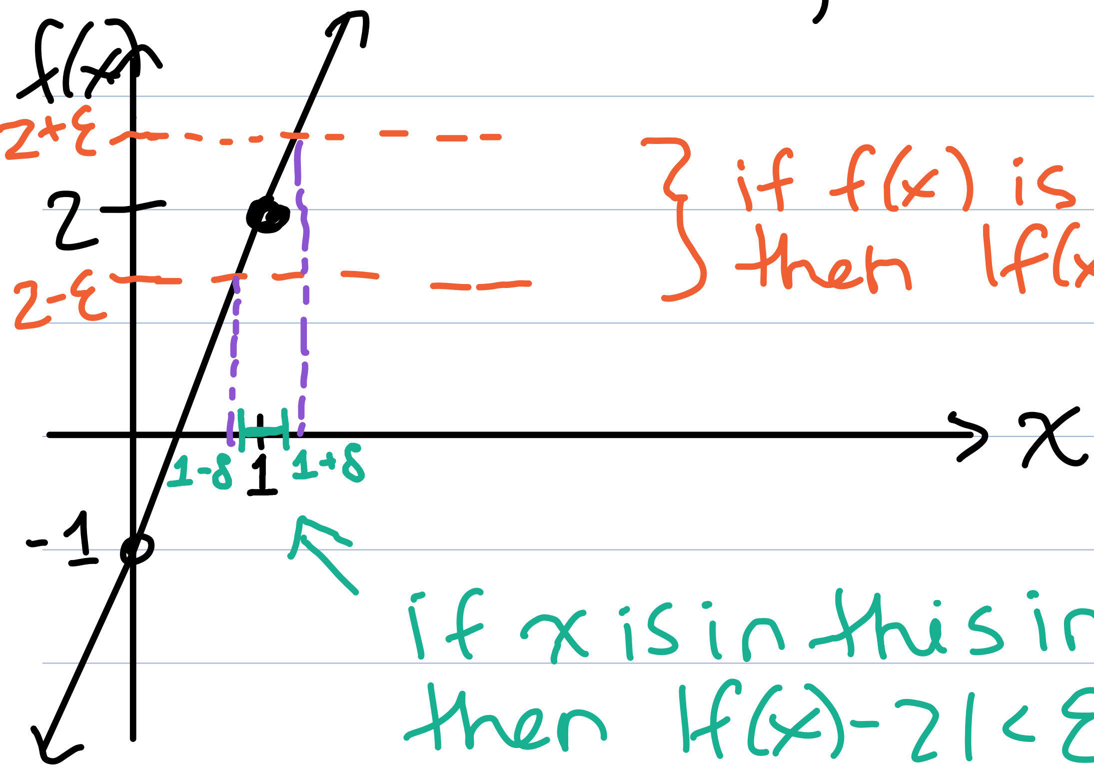
$$\lim_{x \to a} f(x) = L$$

$$\Updownarrow$$

[$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in X$ with $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.]

$(*)$

Ex: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=3x-1$



if $f(x)$ is in this interval, then $|f(x)-2|<\varepsilon$

if x is in this interval, then $|f(x)-2|<\varepsilon$

Guess: $\lim_{x\to 1} f(x) = 2$

Last time, we showed this using sequences defn. Now, we show via ε-δ characterization.

Pf: Fix $\varepsilon>0$ arbitrary.

Scratch:

$$|f(x)-2|<\varepsilon \iff |3x-3|<\varepsilon$$
$$\iff 3|x-1|<\varepsilon$$
$$\iff |x-1|<\varepsilon/3$$

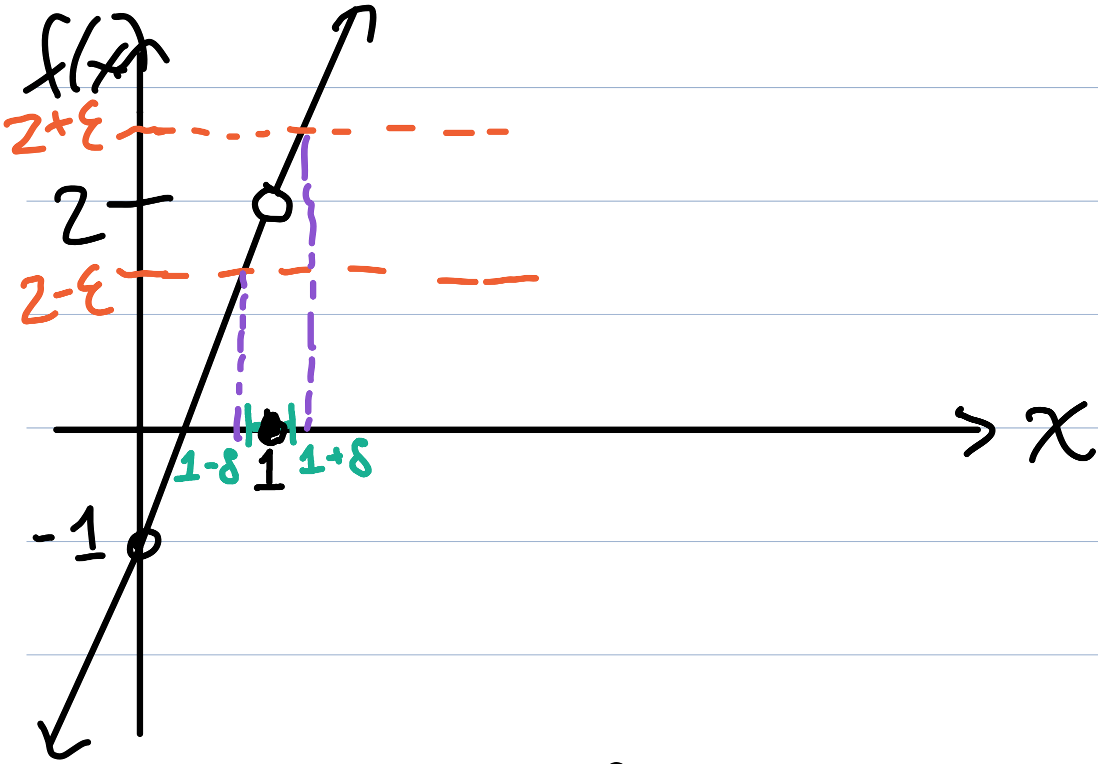
Let $\delta=\varepsilon/3$. Then $0<|x-1|<\delta$ ensures $|f(x)-2|<\varepsilon$.

Alt Ex: $f:\mathbb{R}\to\mathbb{R}$

Remark: If the derivative of f at a is larger, δ must be smaller.

Alt Ex:

$f: \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \begin{cases} 3x-1, & x \neq 1 \\ 0 & x = 1 \end{cases}$



Guess: $\lim_{x \to 1} f(x) = 2$

31 Limit Theorems for Functions

First: just like we combined sequences to get new sequences, we can combine fns to get new fns via pointwise operations

Def: Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$, $c \in \mathbb{R}$

(i) $|f|(x) = |f(x)|$

(ii) $(cf)(x) = cf(x)$

(iii) $(f+g)(x) = f(x) + g(x)$

(iv) $(fg)(x) = f(x)g(x)$

$\forall x \in X$

(v) $(f/g)(x) = \dfrac{f(x)}{g(x)}$ $\forall x \in X$ s.t. $g(x) \neq 0$

Now, we have analogies of limit theorems...

Thm: Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$, $c \in \mathbb{R}$, a is an acc of X, if $\lim_{x \to a} f(x) = L \in \mathbb{R}$ and $\lim_{x \to a} g(x) = M \in \mathbb{R}$,

then

(i) $\lim_{x \to a} |f|(x) = |L|$

(ii) $\lim_{x \to a} (cf)(x) = cL$

(iii) $\lim_{x \to a} (f+g)(x) = L + M$

(iv) $\lim_{x \to a} (fg)(x) = LM$

(v) $\lim_{x \to a} (f/g)(x) = \frac{L}{M}$, as long as $M \neq 0$.

Pf: We will show (iii).

Fix arbitrary $x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \to a$. Then $f(x_n) \to L$ and $g(x_n) \to M$. Thus,

$$\lim_{n\to\infty} f(x_n) + g(x_n) = L + M.$$

This shows $\lim_{x \to a} (f+g)(x) = L + M$.

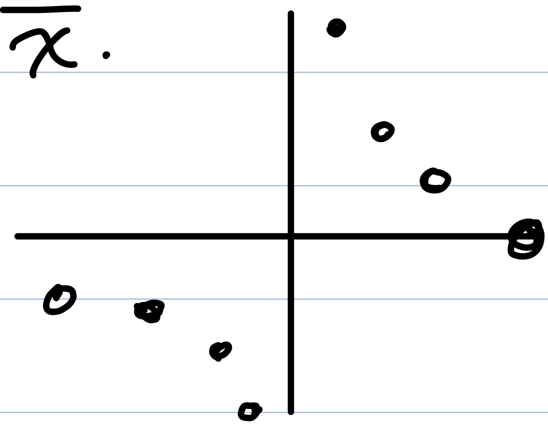
32 One-Sided and Infinite Limits

Recall:

Ex: $X = \{ \frac{1}{m} : m \in \mathbb{Z}, m \neq 0 \}$

$f : X \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

$\lim_{x \to 0} f(x)$ D.N.E.



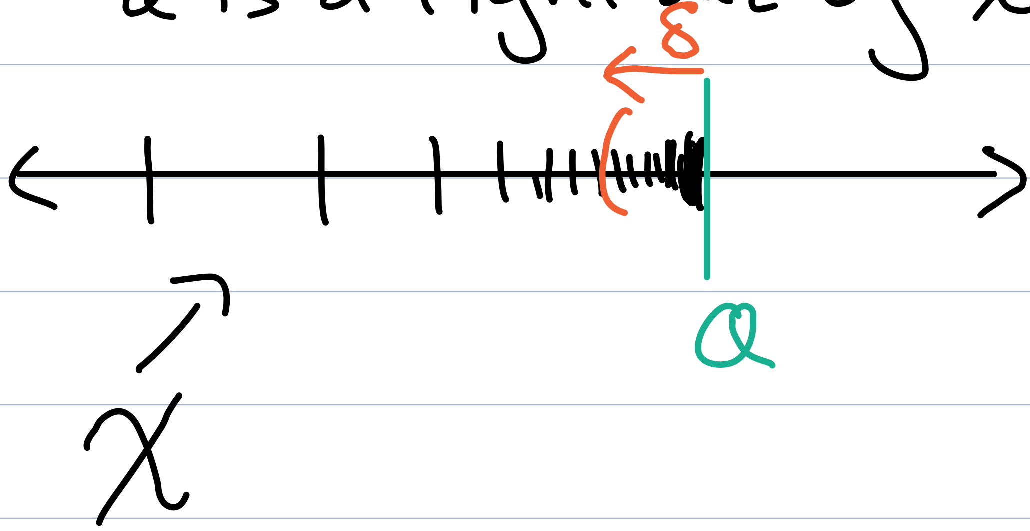
Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$ is

a $\begin{cases} \text{right acc point of } X \\ \text{left acc point of } X \end{cases}$

if $\forall \delta > 0$, $\exists\, x \in X$ s.t.

$$\begin{cases} 0 < a - x < \delta \\ 0 < x - a < \delta \end{cases}$$

α is a right acc of X



<u>Lemma</u>: α is a right (resp. left) acc point of $X \subseteq \mathbb{R}$

$$\Updownarrow$$

$\exists\ x_n : \mathbb{N} \to X \setminus \{\alpha\}$ s.t. $x_n \nearrow \alpha$

$x_n \searrow \alpha$

<u>Pf</u>: Suppose α is a right acc point of X. We define x_n inductively.

Choose x_1 s.t. $x_1 \in X$ and $0 < a - x_1 < 1$.

Suppose we have chosen $x_n \in X \setminus \{a\}$ s.t. $x_{n-1} \leq x_n$ and $0 < a - x_n < \frac{1}{n}$.

Let $\delta = \min\{\frac{1}{n+1}, a - x_n\}$.
By defn of right acc point, $\exists$ $x_{n+1} \in X \setminus \{a\}$ s.t.
$0 < a - x_{n+1} < \delta$.

Since $a - x_{n+1} < a - x_n$, $x_n < x_{n+1}$.
Likewise $a - x_{n+1} < \frac{1}{n+1}$.

This defines an increasing sequence satisfying

$$0 < a - x_n < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

By Squeeze Lemma, $\lim_{n \to \infty} x_n = a$.

Other direction next time.

33 Continuity