Office Hours:
Wed 3:30-4:30pm, Thur 1-2pm

This Friday, last makeup lecture 3:30-4:45pm

3.0 Definition of the Limit of a Function

Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $a$ is an accumulation point of $X$ if for all $\delta > 0$, there exists $x \in X$ such that $0 < |x - a| < \delta$

Lemma: $a$ is an accumulation point of $X \subseteq \mathbb{R}$ if and only if $\exists \{x_n\} : N \rightarrow \mathbb{R}$ such that $x_n \in X \setminus \{a\}$ for all $n \in N$ and $x_n \rightarrow a$. 
**Def:** Given \( X \subseteq \mathbb{R} \) nonempty, \( f: X \rightarrow \mathbb{R} \), \( a \) an accumulation point of \( X \), \( L \in \mathbb{R} \), the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \) if, for all sequences \( \{x_n\} \in X \setminus \{a\} \) s.t. \( x_n \to a \), we have \( \lim_{n \to \infty} f(x_n) = L \).

We denote this as \( \lim_{x \to a} f(x) = L \).

**Ex:** \( X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \)

\( f: X \rightarrow \mathbb{R}, \quad f(x) = \frac{1}{x} \)

\( a = 0 \) is an accumulation point of \( X \)

\( \lim_{x \to 0} f(x) = +\infty \)
Let's prove this!

Fix \( x_n : \mathbb{N} \to X \), s.t. \( x_n \to 0 \).

We must show \( \lim_{n \to \infty} f(x_n) = +\infty \).

Fix \( M \in \mathbb{R} \).

\\

Scratch: WLOG \( M > 0 \).

\[
\frac{1}{x_n} = f(x_n) > M \iff \frac{1}{M} > x_n
\]

If \( M \leq 0 \), then \( f(x_n) = M \) \( \forall n \in \mathbb{N} \)

and we are done. Suppose \( M > 0 \).

Since \( x_n \to 0 \), \( \exists N \in \mathbb{N} \) s.t. \( n > N \)

ensures \( x_n = |x_n - 0| < \frac{1}{M} \).

Thus \( n > N \), \( f(x_n) = \frac{1}{x_n} > M \).

Hence \( \lim_{n \to \infty} f(x_n) = +\infty \).
\[ X = \{ \frac{1}{m} : m \in \mathbb{Z}, m \neq 0 \} \]
\[ f : X \to \mathbb{R}, \quad f(x) = \frac{1}{x}. \]
\[ \lim_{x \to 0} f(x) \quad \text{D.N.E.} \]

**Prop:** Given \( X = \mathbb{R} \) nonempty, \( f : X \to \mathbb{R} \), \( a \) an accumulation point of \( X \), and \( L \in \mathbb{R} \), then
\[ \lim_{x \to a} f(x) = L \]

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X \text{ with } 0 < |x - a| < \delta, \text{ we have } |f(x) - L| < \varepsilon. \]
Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - 1$

If $x$ is in this interval, then $|f(x) - 2| < 3$

Guess: $\lim_{{x \to 1}} f(x) = 2$

Last time, we showed this using sequences defn. Now, we shall via $\varepsilon$-$\delta$ characterization.

Proof: Fix $\varepsilon > 0$ arbitrary.
Scratch:
\[
|f(x) - 2| < 3 \iff |3x - 3| < 3 \\
\iff 3|1x - 1| < 3 \\
\iff |x - 1| < \frac{3}{3}
\]

Let \( \delta = \frac{3}{3} \). Then \( 0 < |x - 1| < \delta \) ensures \( |f(x) - 2| < 3 \).

**Remark:** If the derivative of \( f \) at \( a \) is larger, \( \delta \) must be smaller.
Calculus:
\[ f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 3x + 1, & x \neq 1 \\ 0, & x = 1 \end{cases} \]

**Guess:** \( \lim_{x \to 1} f(x) = 2 \)
First: just like we combined sequences to get new sequences, we can combine fns to get new fns via pointwise operations.

Def: Given $X \subseteq \mathbb{R}$, $f,g : X \rightarrow \mathbb{R}$, $c \in \mathbb{R}$,

(i) $|f| (x) = |f(x)|$

(ii) $(cf)(x) = cf(x)$  \hspace{1cm} \forall x \in X$

(iii) $(f+g)(x) = f(x) + g(x)$

(iv) $(fg)(x) = f(x)g(x)$

(v) $(f/g)(x) = \begin{cases} f(x) & \text{if } g(x) \neq 0 \\ \frac{1}{g(x)} & \text{if } g(x) = 0 \end{cases}$
Now, we have analogies of limit theorems...

**Thm:** Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$, $c \in \mathbb{R}$, $a$ is an acc of $X$, if \( \lim f(x) = L \in \mathbb{R} \) and \( \lim g(x) = M \in \mathbb{R} \), \( x \to a \) then

(i) \( \lim_{x \to a} |f(x)| = |L| \)

(ii) \( \lim_{x \to a} (cf)(x) = cL \)

(iii) \( \lim_{x \to a} (f + g)(x) = L + M \)

(iv) \( \lim_{x \to a} (fg)(x) = LM \)

(v) \( \lim_{x \to a} \frac{f}{g}(x) = \frac{L}{M} \), as long as \( M \neq 0 \).
Pf: We will show (iii).

Fix arbitrary $x_n: \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \to a.$ Then $f(x_n) \to L$ and $g(x_n) \to M.$ Thus,

$$\lim_{n \to \infty} f(x_n) + g(x_n) = L + M.$$ 

This shows $\lim_{x \to a} (f + g)(x) = L + M.$
Recall:

\( \exists x : x = \left\{ \frac{1}{m} : m \in \mathbb{Z}, m \neq 0 \right\} \)

\( f : X \to \mathbb{R}, \ f(x) = \frac{1}{x} \).

\[ \lim_{x \to 0} f(x) \text{ D.N.E.} \]

Def: Given \( X = \mathbb{R}, a \in \mathbb{R} \) is

a right acc point of \( X \)

if for every \( \delta > 0 \), there exists \( x \in X \) s.t.

\[
\begin{align*}
0 < x - a < \delta \\
0 < a - x < \delta
\end{align*}
\]
Lemma: \( \alpha \) is a right (resp. left) acc point of \( X \subseteq \mathbb{R} \).

\[ \exists x_n : \mathbb{N} \rightarrow X \setminus \{ \alpha \} \text{ s.t. } x_n \uparrow \alpha \]

Proof: Suppose \( \alpha \) is a right acc point of \( X \). We define \( x_n \) inductively.
Choose $x_1$ s.t. $x_1 \in X$ and $0 < a - x_1 < 1$.

Suppose we have chosen $x_n \in X \setminus \{a\}$ s.t. $x_{n-1} \leq x_n$ and $0 < a - x_n < \frac{1}{n}$.

Let $\delta = \min \left\{ \frac{1}{n+1}, a - x_n \right\}$. By defn of right acc point, $\exists x_{n+1} \in X \setminus \{a\}$ s.t. $0 < a - x_{n+1} < \delta$.

Since $a - x_{n+1} < a - x_n$, $x_n < x_{n+1}$. Likewise $a - x_{n+1} < \frac{1}{n+1}$.
This defines an increasing sequence satisfying
\[ 0 < a - x_n < \frac{1}{n} \] \[ \forall n \in \mathbb{N}. \]

By Squeeze Lemma, \( \lim_{n \to \infty} x_n = a. \)

Other direction next time.
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