Lecture 12

Office Hours:
Wed 3:30-4:30pm, Thur 1-2pm

This Friday, last makeup lecture 3:30-4:45pm

3.1 Limit Theorems for Functions

Def: Given \( X \subseteq \mathbb{R}, f, g : X \to \mathbb{R} \),

\( c \in \mathbb{R} \)

(i) \( f(1(x)) = |f(x)| \)

(ii) \( (cf)(x) = cf(x) \) \( \forall x \in X \)

(iii) \( (f + g)(x) = f(x) + g(x) \)

(iv) \( (fg)(x) = f(x)g(x) \)

(v) \( \frac{f}{g}(x) = \frac{f(x)}{g(x)} \) \( \forall x \in X \) s.t. \( g(x) \neq 0 \)
Thm: Given \( x \subseteq \mathbb{R}, \ f, g: x \rightarrow \mathbb{R} \), \( c \in \mathbb{R} \), \( \alpha \) is an acc of \( x \), if \( \lim f(x) = L \in \mathbb{R} \) and \( \lim g(x) = M \in \mathbb{R} \), then:

(i) \( \lim_{x \rightarrow \alpha} f(x) = L \)

(ii) \( \lim_{x \rightarrow \alpha} (cf)(x) = cL \)

(iii) \( \lim_{x \rightarrow \alpha} (f + g)(x) = L + M \)

(iv) \( \lim_{x \rightarrow \alpha} (fg)(x) = LM \)

(v) \( \lim_{x \rightarrow \alpha} \left( \frac{f}{g} \right)(x) = \frac{L}{M} \), as long as \( M \neq 0 \).
Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$ is a right acc point of $X$ if $\forall \delta > 0$, $\exists x \in X$ s.t.

\begin{align*}
0 < a - x &< \delta \\
0 < x - a &< \delta
\end{align*}

$a$ is a right acc of $X$
Lemma: $\alpha$ is a right (resp. left) acc point of $X \subseteq \mathbb{R}$

\[ \exists \ x_n \in \mathbb{N} \rightarrow X \setminus \{ \alpha \} \text{ s.t. } x_n \uparrow \alpha \ x_n \downarrow \alpha \]

Proof: We will prove for right acc pts. Last time, we showed $\downarrow$.

Now, we will show $\uparrow$. Fix arbitrary $\delta > 0$. Note that $x_n < \alpha$ \( \forall n \in \mathbb{N} \). To see this, assume for the sake of contradiction, that $x_N \geq \alpha$ for some $N \in \mathbb{N}$. Since $x_N \in X \setminus \{ \alpha \}$ we have $x_N > \alpha$. Let $\varepsilon = x_N - \alpha$.\}
Then, \( \forall n \geq N, \ x_n \geq x_N > a \).
Hence \( |x_n - a| > 3 \) \( \forall n \geq N \).
Thus \( x_n \not\to a \), which is a contradiction.

Since \( x_n \to a \), \( \exists N \ s.t. \)
\[ |x_n - a| < \varepsilon \iff a - x_n < \varepsilon. \]

\[ \square \]

**Def:** Given \( X \subseteq \mathbb{R}, f: X \to \mathbb{R} \),
\( a \in \{ \text{right acc pt of } X, \text{let } \mathbb{R} \} \)
the limit of \( f(x) \) as \( x \) approaches \( a \) \{ from the left \}
is \( L \) if \{ from the right \}
\( \forall \ x_n : |N \to X \setminus \{a\} \ s.t. \ \{x_n \to a, \lim_{n \to \infty} f(x_n) = L \} \)

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We denote this as:

\[ \lim_{x \to a^-} f(x) = L \]

\[ \lim_{x \to a^+} f(x) = L \]

Ex:

![Graph showing a limit approaching a point from both sides](image)

Q: \( a \) is an acc pt of \( f \)

or

\( a \) is a R and L acc pt of \( f \)

A: HW7
Ex: Consider \( X = \{ \frac{1}{n} : n \in \mathbb{N} \} \). Then \( a = 0 \) is a acc point, a L acc point, but not a R acc pt.

Thm: Suppose \( a \) is a R and L acc pt of \( X \). Then

\[
\lim_{x \to a^-} f(x) = L \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.
\]

Recall: Given \( x_n : \mathbb{N} \to \mathbb{R} \),
- \( x_n \) converges to \( L \in \mathbb{R} \)
- \( \uparrow \)
- every subsequence \( x_{n_k} \) has a further subsequence \( x_{n_{k_l}} \) s.t. \( x_{n_{k_l}} \to L \).
Note that \( \Rightarrow \) is immediate from defn.

Now, we will show \( \Leftarrow \).

We will show in the case that \( L \in \mathbb{R} \). For the case \( L = \pm \infty \), HW7.

Fix \( x_n : \mathbb{N} \to X \) s.t. \( x_n \to \alpha \).

We must show \( \lim_{n \to \infty} f(x_n) = L \).

Let \( x_{n_k} \) be an arb subsequence of \( x_n \). Then there exists a further subseq \( x_{n_{k_k}} \) that is monotone, so either \( x_{n_{k_k}} \uparrow \alpha \) or \( x_{n_{k_k}} \downarrow \alpha \).
In either case, we have
\[ \lim_{l \to \infty} f(x_{n_{k_l}}) = L. \]

This shows that \( f(x_n) : \mathbb{N} \to \mathbb{R} \) has the property that every subseq \( f(x_{n_{k_l}}) \) has a further subseq \( f(x_{n_{k_{l_2}}}) \) s.t. \( f(x_{n_{k_{l_2}}}) \to L. \)

Thus, \( \lim_{n \to \infty} f(x_n) = L. \)

Last type of limiting behavior: as \( x \to \pm \infty \).
Intuitively, \( +\infty \) behaves like an \( \infty \) for \( f(x) \) if \( X \) is unbounded above.

**Def:** Given \( X \subseteq \mathbb{R} \) \( \{ \) unbounded above, unbounded below \( \} \), \( f: X \to \mathbb{R}, L \in \mathbb{R} \), then the limit of \( f(x) \) as \( x \) approaches \( +\infty \) \( -\infty \) is \( L \) if, \( \forall x_n : \mathbb{N} \to X \) with

\[
\begin{align*}
\lim_{n \to \infty} x_n &= +\infty \\
\lim_{n \to \infty} x_n &= -\infty
\end{align*}
\]
we have \( \lim_{n \to \infty} f(x_n) = L \).

If this holds, we write \( \lim_{x \to \infty} f(x) = L \) \( \lim_{x \to -\infty} f(x) = L \).
Ex: \( f: (0, +\infty) \to \mathbb{R} \) \( f(x) = \frac{\sin x}{x} \)

We expect \( \lim_{x \to +\infty} f(x) = 0 \).

pf: Fix \( x_n: \mathbb{N} \to (0, +\infty) \) s.t. \( \lim_{n \to \infty} x_n = +\infty \).

We must show \( \lim_{n \to \infty} f(x_n) = 0 \).

Note that \( -\frac{1}{x_n} \leq f(x_n) \leq \frac{1}{x_n} \).

Since \( -\frac{1}{x_n}, \frac{1}{x_n} \to 0 \), the result follows from Squeeze Lemma.

33 Continuity
**Def:** Given \( f: \mathbb{R} \rightarrow \mathbb{R} \), \( \alpha \in \mathbb{R} \), \( f \) is continuous at \( \alpha \) if either

1. \( \alpha \) is an accumulation point of \( \mathbb{R} \) and \( \lim_{x \to \alpha} f(x) = f(\alpha) \)
2. \( \alpha \) is an isolated point

**Ex:**

\[ f(x) = c_1 x^n + c_2 x^{n-1} + \ldots + c_n x + c_{n+1} \]

\( f \) is continuous at \( \alpha \), \( \forall \alpha \in \mathbb{R} \)
Thm: Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$ cts at $a \in X$, then the following are cts at $a$:

(i) $|f|$
(ii) $cf$, for $c \in \mathbb{R}$
(iii) $f + g$
(iv) $fg$
(v) $f \circ g$, provided $g(a) \neq 0$.

Pf: We will show (v). If $a$ is an isolated pt wrt $X$, the result is immediate. Assume $a$ is an acc point of $X$. Let

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M.$$ 

By assumption that $f$ and $g$ are cts at $a$. 
By earlier theorem,
\[
\lim_{{x \to a}} \frac{f}{g}(x) = \frac{f(a)}{g(a)} = \frac{f(a)}{g(a)} \cdot \frac{\xi}{\xi} = (f/g)(a).
\]

**Thm (3-8 char of continuity):** Given \( X \in \mathbb{R}, f : X \to \mathbb{R}, \) and \( f \) is continuous at \( a \) if

\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } x \in X \text{ and } |x - a| < \delta \text{ ensures } |f(x) - f(a)| < \varepsilon.
\]
\[ \text{pf: Suppose } f \text{ is cts at } a. \]
\[ \text{Fix } \epsilon > 0 \text{ arbitrary.} \]

If \( a \) is an isolated point w.r.t. \( X \), then \( a \) is not an accpt of \( X \), so \( \exists \delta > 0 \text{ s.t. } x \in X \text{ and } |x-a| < \delta \text{ ensures } x = a. \]

Thus: \( |f(x) - f(a)| = 0 < \epsilon. \)

Now, suppose \( a \) is an accpt of \( X \). Then, since \( f \) is cts at \( a \),
\[ \lim_{x \to a} f(x) = f(a). \]

Last time, we showed that this implies \( \exists \delta > 0 \text{ s.t. } x \in X \text{ and } 0 < |x-a| < \delta \text{ ensures } |f(x) - f(a)| < \epsilon. \)
This shows $\forall x \in X, |x-a| < \delta$ ensures $|f(x) - f(a)| < \epsilon$.

Next time: other implication.

34 The Heine-Borel Theorem