Lecture 12

Office Hours:

Wed 3:30-4:30pm, Thur 1-2pm

This Friday, last makeup lecture 3:30-4:45pm

31 Limit Theorems for Functions

Def: Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$, $c \in \mathbb{R}$

$$\left.\begin{array}{l} \text{(i) } |f|(x) = |f(x)| \\ \text{(ii) } (cf)(x) = cf(x) \\ \text{(iii) } (f+g)(x) = f(x) + g(x) \\ \text{(iv) } (fg)(x) = f(x)g(x) \end{array}\right\} \forall x \in X$$

(v) $\left(f/g\right)(x) = \dfrac{f(x)}{g(x)}$ $\forall x \in X$ s.t. $g(x) \neq 0$

Thm: Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$, $c \in \mathbb{R}$, a is an acc of X, if $\lim_{x \to a} f(x) = L \in \mathbb{R}$ and $\lim_{x \to a} g(x) = M \in \mathbb{R}$,

then

(i) $\lim_{x \to a} |f|(x) = |L|$

(ii) $\lim_{x \to a} (cf)(x) = cL$

(iii) $\lim_{x \to a} (f+g)(x) = L + M$

(iv) $\lim_{x \to a} (fg)(x) = LM$

(v) $\lim_{x \to a} (f/g)(x) = \frac{L}{M}$, as long as $M \neq 0$.

32 One-Sided and Infinite Limits

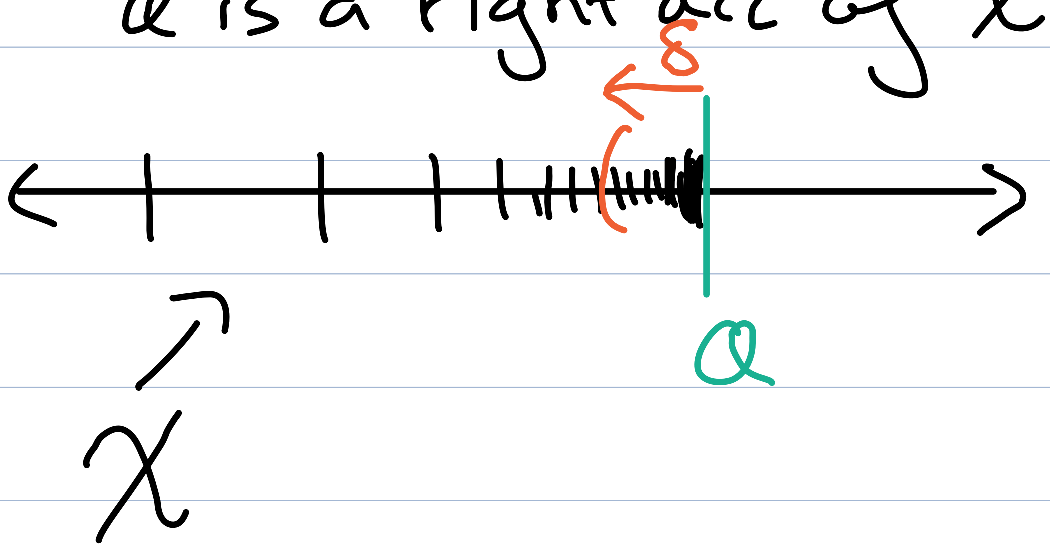
Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$ is

a $\begin{cases} \text{right acc point of } X \\ \text{left acc point of } X \end{cases}$

if $\forall \delta > 0$, $\exists\, x \in X$ s.t.

$$\begin{cases} 0 < a - x < \delta \\ 0 < x - a < \delta \end{cases}$$

a is a right acc of X



Lemma: a is a right (resp. left) acc point of $X \subseteq \mathbb{R}$

$\Updownarrow$

$\exists\, x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \nearrow a$
$x_n \searrow a$

Pf: We will prove for right acc pts.
Last time, we showed $\Downarrow$.

Now, we will show $\Uparrow$. Fix arbitrary $\delta > 0$. Note that $x_n < a \ \forall n \in \mathbb{N}$. To see this, assume for the sake of contradiction, that $x_N \geq a$ for some $N \in \mathbb{N}$. Since $x_N \in X \setminus \{a\}$, we have $x_N > a$. Let $\varepsilon = x_N - a$.

Then, $\forall\, n \geq N$, $x_n \geq x_N > a$.
Hence $|x_n - a| > \varepsilon$ $\forall\, n \geq N$.
Thus $x_n \not\to a$, which is a contradiction.

Since $x_n \to a$, $\exists\, N$ s.t.
$$|x_N - a| < \delta \overset{x_N < a}{\Longleftrightarrow} a - x_N < \delta.$$
$\square$

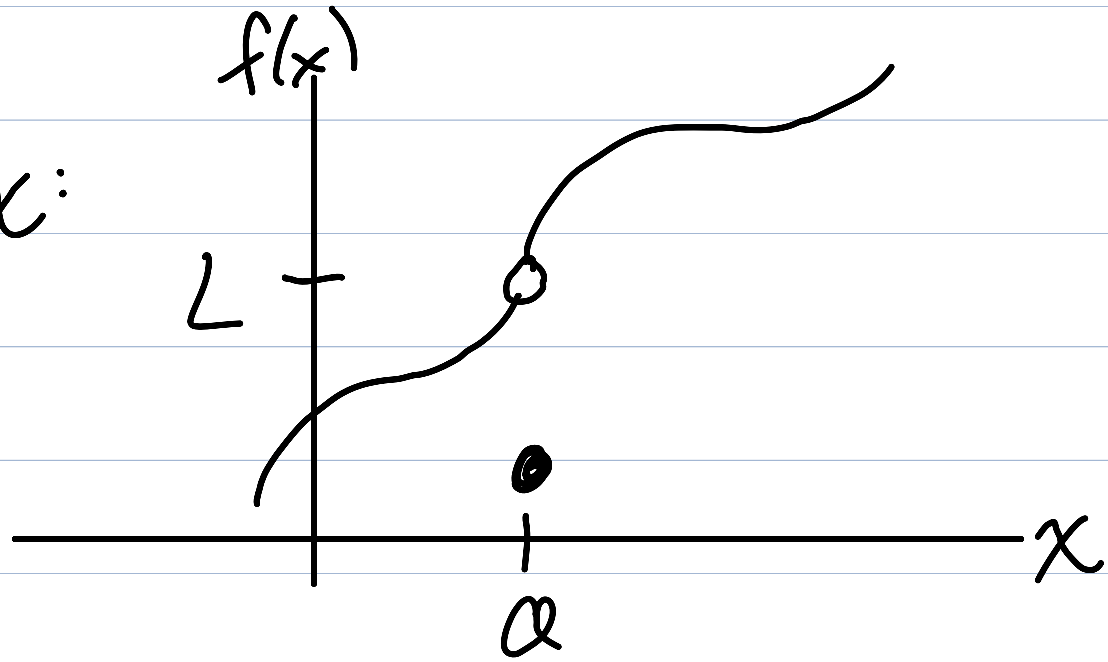
Def: Given $X \subseteq \mathbb{R}$, $f: X \to \overline{\mathbb{R}}$, a a $\begin{cases} \text{right acc pt} \\ \text{left acc pt} \end{cases}$ of X, $L \in \overline{\mathbb{R}}$, the limit of $f(x)$ as x approaches a $\begin{cases} \text{from the left} \\ \text{from the right} \end{cases}$ is L if $\forall\, x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $\begin{cases} x_n \nearrow a \\ x_n \searrow a \end{cases}$, $\lim_{n\to\infty} f(x_n) = L$.

We denote this as

$$\lim_{x \to a^-} f(x) = L$$

$$\lim_{x \to a^+} f(x) = L$$

Ex:



Q: a is an acc pt of X

$\Updownarrow$

a is a R ~~and~~ or L acc pt of X

A: HW7

Ex: Consider $X = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then $a=0$ is a acc point,
a L acc point, but not a R acc pt.

<u>Thm</u>: Suppose a is a R and L acc pt of X. Then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

Recall: Given $x_n : \mathbb{N} \to \mathbb{R}$,

- x_n converges to $L \in \mathbb{R}$

$\Updownarrow$

- every subsequence x_{n_k} has a further subsequence $x_{n_{k_\ell}}$ s.t. $x_{n_{k_\ell}} \to L$.

Pf:

Note that $\Rightarrow$ is immediate from defn.

Now, we will show $\Leftarrow$.

We will show in the case that $L \in \mathbb{R}$. For the case $L = \pm\infty$, HW7.

Fix $x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \to a$. We must show $\lim_{n\to\infty} f(x_n) = L$.

Let x_{n_k} be an arb subsequence of x_n. Then there exists a further subseq $x_{n_{k_\ell}}$ that is monotone, so either $x_{n_{k_\ell}} \nearrow a$ or $x_{n_{k_\ell}} \searrow a$.

In either case, we have
$$\lim_{\ell \to \infty} f(x_{n_{k_\ell}}) = L.$$

This shows that $f(x_n): \mathbb{N} \to \mathbb{R}$ has the property that every subseq $f(x_{n_k})$ has a further subseq $f(x_{n_{k_\ell}})$ s.t. $f(x_{n_{k_\ell}}) \to L$.

Thus, $\lim_{n \to \infty} f(x_n) = L$. $\square$

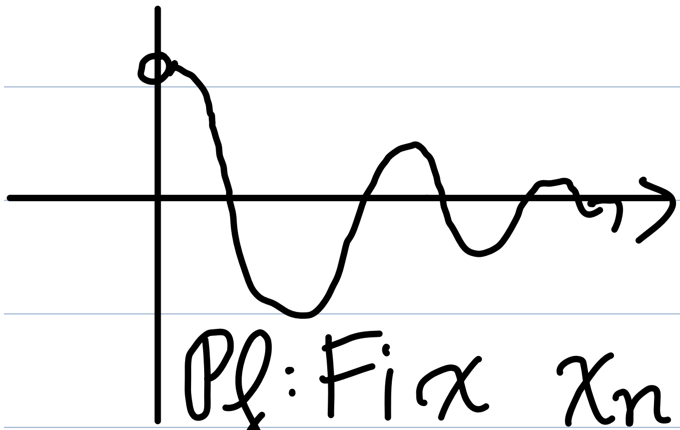
Last type of limiting behavior: as $x \to \pm\infty$.

Intuitively, $+\infty$ behaves like an acc for X if X is unbounded above.

<u>Def</u>: Given $X \subseteq \mathbb{R}$ $\begin{cases} \text{unbounded above,} \\ \text{unbounded below} \end{cases}$ $f: X \to \overline{\mathbb{R}}$, $L \in \overline{\mathbb{R}}$, then the limit of $f(x)$ as x approaches $\begin{cases} +\infty \\ -\infty \end{cases}$ is L if, $\forall\, x_n : \mathbb{N} \to X$ with $\begin{cases} \lim\limits_{n\to\infty} x_n = +\infty \\ \lim\limits_{n\to\infty} x_n = -\infty \end{cases}$ we have $\lim\limits_{n\to\infty} f(x_n) = L$.

If this holds, we write $\begin{cases} \lim\limits_{x\to\infty} f(x) = L \\ \lim\limits_{x\to-\infty} f(x) = L \end{cases}$

Ex: $f:(0,+\infty)\to\mathbb{R}$ $f(x)=\dfrac{\sin(x)}{x}$



We expect $\lim\limits_{x\to+\infty} f(x)=0$.

Pf: Fix $x_n:\mathbb{N}\to(0,+\infty)$ s.t. $\lim\limits_{n\to\infty} x_n=+\infty$.

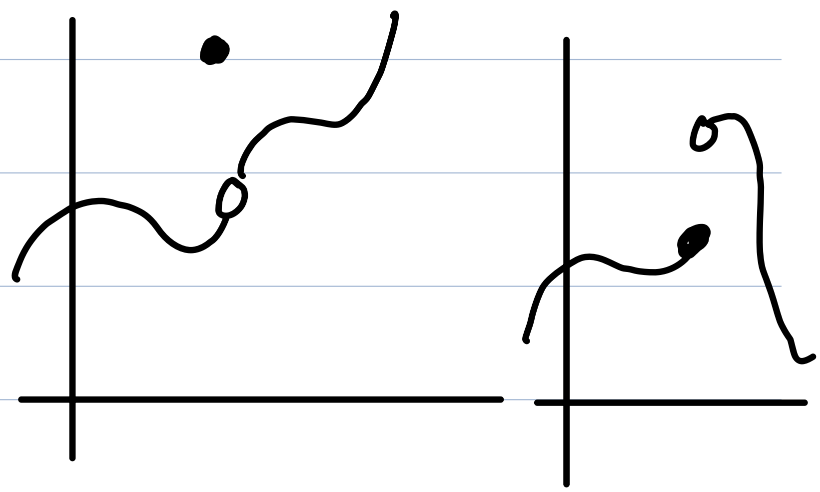
We must show $\lim\limits_{n\to\infty} f(x_n)=0$.

Note that $\dfrac{-1}{x_n}\le f(x_n)\le\dfrac{1}{x_n}$.

Since $\frac{-1}{x_n}, \frac{1}{x_n}\to 0$, the result follows from Squeeze Lemma.

33 Continuity

vs

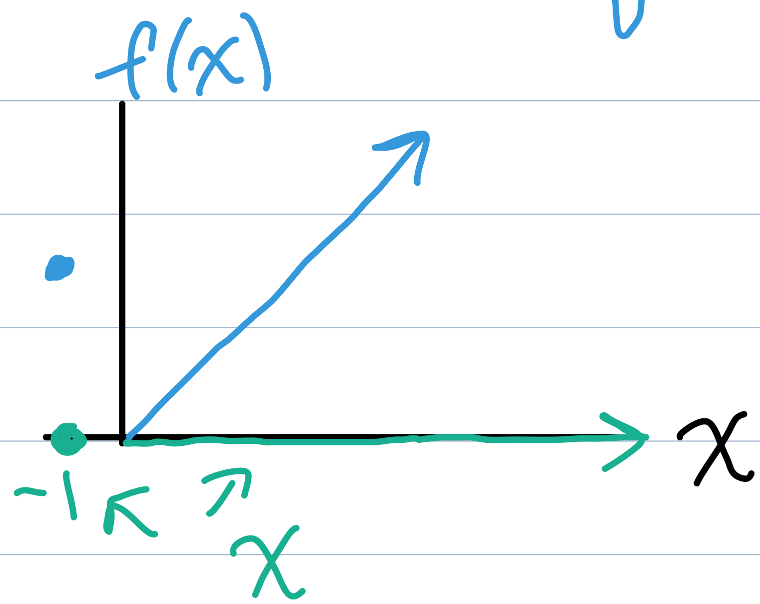


Def: Given $X \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$, $a \in X$, <u>f is continuous at a</u> if either

(i) a is an acc pt of X and $\lim_{x \to a} f(x) = f(a)$

(ii) $\overbrace{a \text{ is not an acc pt of } X}^{a \text{ is an isolated point}}$

Ex:



Ex: $f: \mathbb{R} \to \mathbb{R}$,
$f(x) = c_1 x^n + c_2 x^{n-1} + \dots + c_n x + c_{n+1}$
f is cts at a, $\forall a \in \mathbb{R}$

Thm: Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$ cts at $a \in X$, then the following are cts at a:

(i) $|f|$

(ii) cf, for $c \in \mathbb{R}$

(iii) $f+g$

(iv) fg

(v) f/g, provided $g(a) \neq 0$.

Pf: We will show (v). If a is an isolated pt wrt X, the result is immediate. Assume a is an acc point of X. Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.

$$L = f(a) \qquad M = g(a)$$

By assumption that f and g are cts at a.

By earlier theorem,

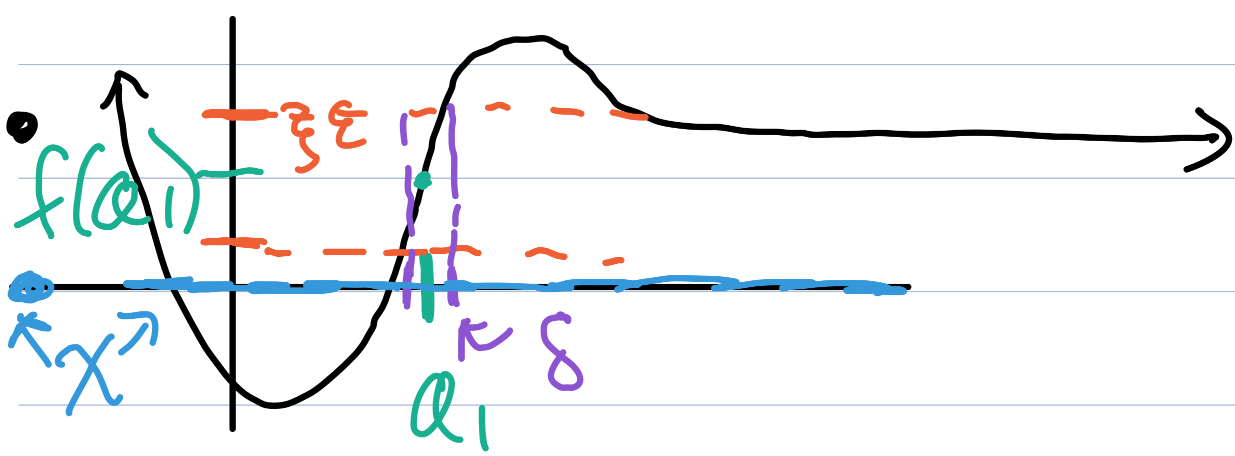
$$\lim_{x \to a} (f/g)(x) = \frac{L}{m} = \frac{f(a)}{g(a)} = (f/g)(a).$$

<u>Thm</u> (ε-δ char of cty): Given $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, $a \in X$

f is cts at a

$\Updownarrow$

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $x \in X$ and $|x - a| < \delta$ ensures $|f(x) - f(a)| < \varepsilon$.



Pf: Suppose f is cts at a.
Fix $\varepsilon > 0$ arbitrary.

If a is an isolated point w.r.t. X,
then a is not an acc pt of X,
so $\exists\ \delta > 0$ s.t. $x \in X$ and
$|x-a| < \delta$ ensures $x = a$.
Thus $|f(x) - f(a)| = 0 < \varepsilon$.

Now, suppose a is an acc point
of X. Then, since f is cts at a,
$\lim_{x \to a} f(x) = f(a)$. Last time, we
showed that this implies $\exists\ \delta > 0$
s.t. $x \in X$ and $0 < |x-a| < \delta$
ensures $|f(x) - f(a)| < \varepsilon$.

This shows $\forall x \in X$,
$|x-a|<\delta$ ensures $|f(x)-f(a)|<\varepsilon$.

Next time: other implication.

34 The Heine-Borel Theorem