Lecture 13

Office Hours:

Wed 3:30-4:30pm, Thur 1-2pm

Midterm 2: Wednesday, May 29th

$\square$

Def: Given $X \subseteq \mathbb{R}$, $f: X \to \overline{\mathbb{R}}$, a $\begin{cases} \text{right acc pt} \\ \text{left acc pt} \end{cases}$ of X, $L \in \overline{\mathbb{R}}$,

the limit of $f(x)$ as x approaches a $\begin{cases} \text{from the left} \\ \text{from the right} \end{cases}$ is L if

$\forall\, x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $\begin{cases} x_n \nearrow a \\ x_n \searrow a \end{cases}$, $\lim_{n \to \infty} f(x_n) = L$.

We denote this as

$$\lim_{x \to a^-} f(x) = L$$

$$\lim_{x \to a^+} f(x) = L$$

Thm: Suppose a is a $\mathbb{R}$ and acc pt of X. Then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

Def: Given $X \subseteq \mathbb{R}$ $\begin{cases} \text{unbounded above,} \\ \text{unbounded below;} \end{cases}$

$f: X \to \overline{\mathbb{R}}$, $L \in \overline{\mathbb{R}}$, then the limit of $f(x)$ as x approaches $\begin{cases} +\infty \\ -\infty \end{cases}$

is L if, $\forall x_n: \mathbb{N} \to X$ with

$$\begin{cases} \lim_{n \to \infty} x_n = +\infty \\ \lim_{n \to \infty} x_n = -\infty \end{cases}$$

we have $\lim_{n \to \infty} f(x_n) = L$.

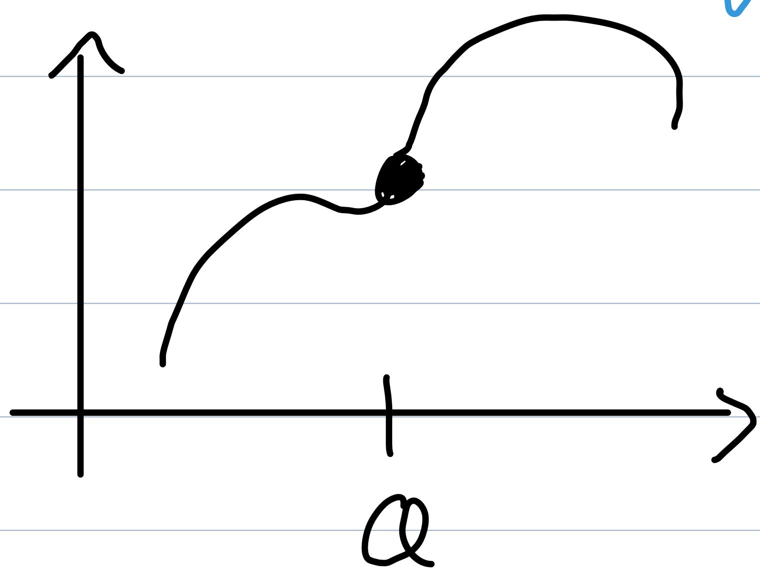
If this holds, we write $\begin{cases} \lim_{x \to \infty} f(x) = L \\ \lim_{x \to -\infty} f(x) = L \end{cases}$

33 Continuity

<u>Def</u>: Given $X \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$, $a \in X$, <u>f is continuous at a</u> if either

(i) a is an acc pt of X and $\lim_{x \to a} f(x) = f(a)$

(ii) $\overbrace{a \text{ is not an acc pt of } X}^{a \text{ is an isolated point}}$



Thm: Given $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$ cts at $a \in X$, then the following are cts at a:

(i) $|f|$

(ii) cf, for $c \in \mathbb{R}$

(iii) $f+g$

(iv) fg

(v) f/g, provided $g(a) \neq 0$.

Thm (ε-δ char of cty): Given $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, $a \in X$

f is cts at a

$\Updownarrow$

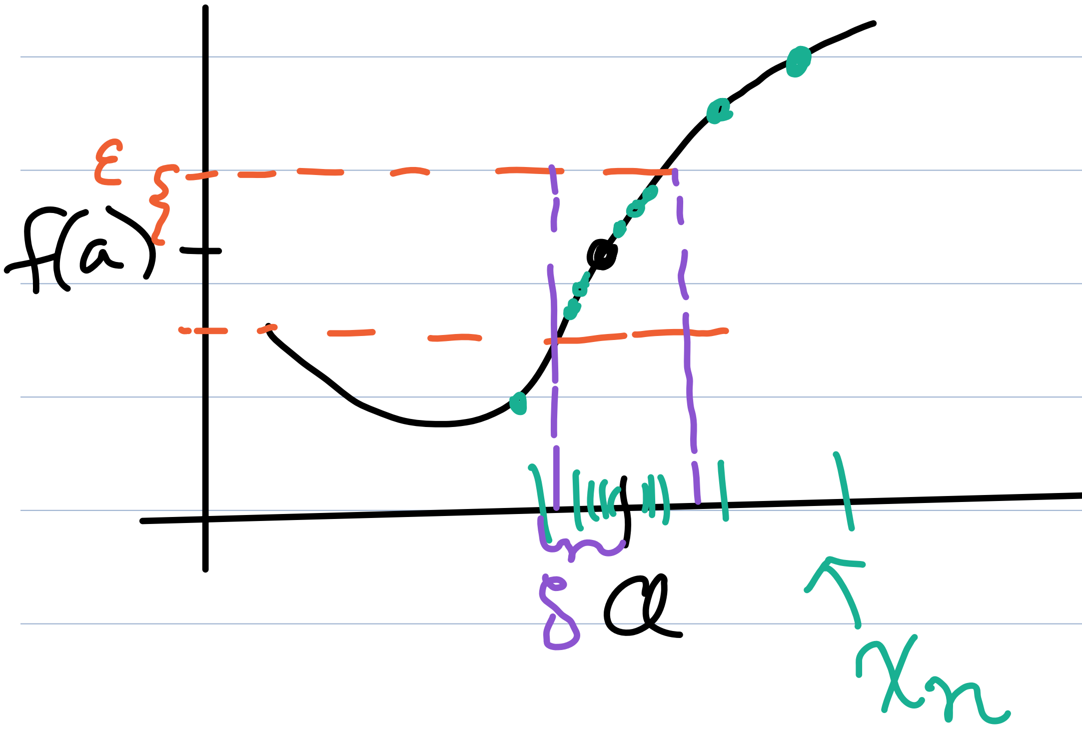
$\left\{ \begin{array}{l} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } x \in X \text{ and} \\ |x-a| < \delta \text{ ensures } |f(x) - f(a)| < \varepsilon. \end{array} \right.$ $(*)$

Pf:
Last time, we showed $\Downarrow$.

Now, we show $\Uparrow$.
Suppose $(*)$ holds.
If a is isolated wrt, there is nothing to show. Suppose a is an acc pt of X.

arbitrary $x_n \to a$.
Fix $x_n : \mathbb{N} \to X \setminus \{a\}$. We must show $\lim_{n\to\infty} f(x_n) = f(a)$.

Fix $\varepsilon > 0$. We must find N s.t. $n \geq N$ ensures $|f(x_n) - f(a)| < \varepsilon$.



By (✱), $\exists\, \delta > 0$ s.t. $x \in X$ and $|x-a| < \delta$ ensures $|f(x)-f(a)| < \varepsilon$.

Since $x_n \to a$, $\exists\, N$ s.t. $n \geq N$ ensures $|x_n - a| < \delta \Rightarrow |f(x_n) - f(a)| < \varepsilon$. $\square$

One last important way to combine functions...

<u>Def</u>: Consider $X, Y \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$, $g: Y \to \mathbb{R}$, $g(Y) \subseteq X$, then define $f \circ g: Y \to \mathbb{R}$ by
$$(f \circ g)(y) = f(g(y)).$$

<u>Thm</u>: Consider $X, Y \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$, $g: Y \to \mathbb{R}$, $g(Y) \subseteq X$. Then for any $a \in Y$, if g is cts at a, f is cts at $g(a)$ we have that $f \circ g$ is cts at a.

Pf: Fix arbitrary $\varepsilon > 0$.
Since f is cts at $g(a)$, $\exists\, \delta_1 > 0$ s.t. $x \in X$, $|x - g(a)| < \delta_1 \Rightarrow |f(x) - f(g(a))| < \varepsilon$.

Since g is cts at a, $\exists\, \delta_2 > 0$ s.t. $y \in Y$, $|y - a| < \delta_2 \Rightarrow |g(y) - g(a)| < \delta_1$.
$\Rightarrow |f(g(y)) - f(g(a))| < \varepsilon$.

$\boxed{\text{34 The Heine-Borel Theorem}}$

Def: $f : [a,b] \to \mathbb{R}$ is cts on $[a,b]$ if it is cts at x, $\forall\, x \in [a,b]$.

Goal: cts fns on a closed interval attain their max and min.

Optimization:

f is cts fn　　compact set, e.g. $[a,b]$

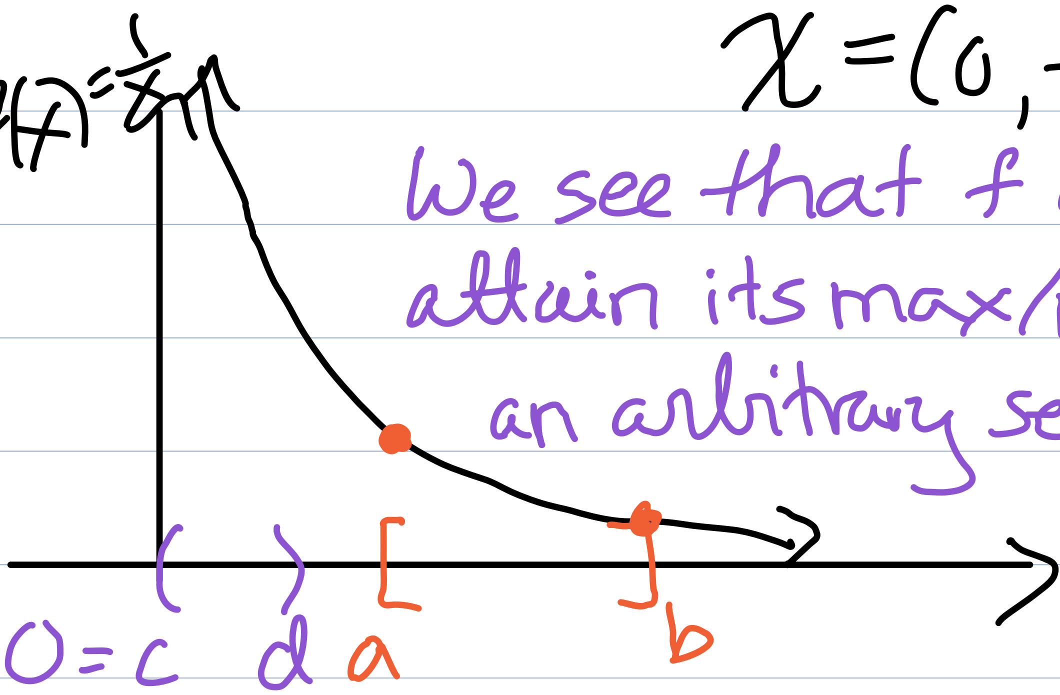
$$\min\{f(x) : x \in C\} = \underbrace{f(x^*)}_{\text{"the minimum of } f \text{ on } C\text{"}}$$

"the minimizer"

we show $\exists\, x^* \in C$ s.t. above equality holds.

Ex: $f(x) = \frac{1}{x}$　　$X = (0, +\infty)$

We see that f will not attain its max/min on an arbitrary set C.



$f : X \to \mathbb{R}$ is continuous, since it is continuous at x, $\forall x \in (0, +\infty)$

Our first step towards this goal will be to show that

$$\{f(x) : x \in [a,b]\}$$

is a bounded subset of $\mathbb{R}$.

Def: Given $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, and $Y \subseteq \mathbb{R}$, f is bounded on Y if $\exists$ $M \in \mathbb{R}$ s.t.

$$|f(x)| \leq M \quad \forall x \in X \cap Y.$$

Lemma: Given $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, if f is cts at $c \in X$, then $\exists$ $\delta > 0$ s.t. f is bdd on $(c-\delta, c+\delta)$.

Pf: Let $\varepsilon = 1$.
Then $\exists\, \delta > 0$ s.t. $x \in X$, $|x-c| < \delta$ ensures $|f(x) - f(c)| < 1$.

Thus $\forall\, x \in X \cap (c-\delta, c+\delta)$,

$$\begin{aligned} |f(x)| &= |f(x) - f(c) + f(c)| \\ &\le |f(x) - f(c)| + |f(c)| \\ &< 1 + |f(c)| \end{aligned}$$ $\square$

Idea behind our strategy of showing $f: [a,b] \to \mathbb{R}$ ~~cts~~ is bdd on $[a,b]$.

By Lemma, $\forall\, c \in [a,b]$, $\exists$ open interval I_c containing c

s.t. f is bdd on I_c.

<u>Claim</u>: If f is bounded on $\{Y_i\}_{i=1}^{n} \subseteq 2^{\mathbb{R}}$, then f is bdd on $\bigcup_{i=1}^{n} Y_i$.

<u>Pf</u>: By hypothesis, $\exists \{M_i\}_{i=1}^{n} \subseteq \mathbb{R}$ s.t. $|f(x)| \leq M_i \ \forall x \in X \cap Y_i$.
Thus $|f(x)| \leq \max\{M_1, \ldots, M_n\} \forall x \in X \cap \left(\bigcup_{i=1}^{n} Y_i\right)$

Problem: We have $[a,b] \subseteq \underbrace{\bigcup_{c \in [a,b]} I_c}$

we don't necessarily know f is bdd on infinite union

Ex: $f(x) = x$

$I_\alpha = (\alpha, \alpha+1)$

f is bdd on I_α $\forall\ \alpha \in \mathbb{N}$

f is not bdd on $\bigcup_{\alpha \in \mathbb{N}} I_\alpha$.

Good news: We actually only need finitely many $\{c_i\}_{i=1}^{n}$ s.t. $[a,b] \subseteq \bigcup_{i=1}^{n} I_{c_i}$.

Thm (Heine-Borel):
Let $\mathcal{J}$ be a collection of open intervals s.t.

$$[a,b] \subseteq \bigcup \mathcal{J}.$$

Then there exists $\{I_1, I_2, \ldots, I_n\} \subseteq \mathcal{J}$ s.t. $[a,b] \subseteq \bigcup_{i=1}^{n} I_i$.

Pf: If $a=b$, the result is immediate. Assume $a<b$.

Let $X = \{x \in (a,b] : [a,x] \subseteq \bigcup_{i=1}^{n} I_i$ for some $\{I_i\}_{i=1}^{n} \subseteq \mathcal{J}\}$

Our goal is to show $b \in X$.

Since $[a,b] \subseteq \bigcup \mathcal{J}$, $\exists$ $I \in \mathcal{J}$ s.t. $a \in I$. Let $(s_0, t_0) := I$. Then $a < t_0$, so $\exists$ x_0 s.t.

$$a < x_0 < \min\{t_0, b\}.$$

Thus $x_0 \in (a,b]$ and $x_0 \in I$, so $[a, x_0] \subseteq I$. Hence $x_0 \in X$, so $X \neq \emptyset$.

Furthermore, $\mathcal{X}$ is bounded above by b. Let $c = \sup(\mathcal{X})$. Note $a \le x_0 \le c$ and $c \le b$. So $c \in [a,b]$, and $\exists \tilde{I} \in \mathcal{G}$ s.t. $c \in \tilde{I}$. Let $(s_1, t_1) := \tilde{I}$.

Since $s_1 < c$, s_1 is not an upper bound of $\mathcal{X}$, so $\exists x_1 \in \mathcal{X}$ s.t. $s_1 < x_1 \le c$. By def $\mathcal{X}$, $\exists \{I_i\}_{i=1}^n \subseteq \mathcal{G}$ s.t.

$$[a, x_1] \subseteq \bigcup_{i=1}^n I_i.$$

So
$$[a, c] \subseteq \left(\bigcup_{i=1}^n I_i\right) \cup \tilde{I}.$$

Thus $c \in \mathcal{X} \Rightarrow c = \max(\mathcal{X})$.

Assume, for the sake of contradiction, $c < b$. $\exists$ d s.t. $c < d < \min\{t_1, b\}$. Then

$$[a, d] \subseteq \left(\bigcup_{i=1}^{n} I_i\right) \cup \tilde{I}.$$

Thus $d \in X$ and $d > \sup(X)$, which is a contradiction. $\square$