Lecture 14

Office Hours:

Wed 3:30-4:30pm, Thur 1-2pm

Midterm 2: Wednesday, May 29th

<u>Def</u>: Consider $X, Y \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$, $g: Y \to \mathbb{R}$, $g(Y) \subseteq X$, then define $f \circ g: Y \to \mathbb{R}$ by
$$(f \circ g)(y) = f(g(y)).$$

<u>Thm</u>: Consider $X, Y \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$, $g: Y \to \mathbb{R}$, $g(Y) \subseteq X$. then for any $a \in Y$, if g is cts at a, f is cts at $g(a)$ we have that $f \circ g$ is cts at a.

34 The Heine-Borel Theorem

<u>Def</u>: $f : [a,b] \to \mathbb{R}$ is cts on $[a,b]$ if it is cts at x, $\forall x \in [a,b]$.

<u>Goal</u>: cts fns on a closed interval attain their max and min.

Our first step towards this goal will be to show that

$$\{f(x) : x \in [a,b]\}$$

is a bounded subset of $\mathbb{R}$.

Def: Given $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, and $Y \subseteq \mathbb{R}$, <u>f is bounded on Y</u> if $\exists\, m \in \mathbb{R}$ s.t.

$$|f(x)| \leq m \quad \forall\, x \in X \cap Y.$$

Lemma: Given $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, if f is cts at $c \in X$, then $\exists\, \delta > 0$ s.t. f is bdd on $(c-\delta, c+\delta)$.

Claim: If f is bounded on $\{Y_i\}_{i=1}^{n} \subseteq 2^{\mathbb{R}}$, then f is bdd on $\bigcup_{i=1}^{n} Y_i$.

Problem: We have $[a,b] \subseteq \bigcup_{c \in [a,b]} I_c$

we don't necessarily know f is bdd on infinite union

Good news: We actually only need finitely many $\{c_i\}_{i=1}^n$ s.t. $[a,b] \subseteq \bigcup_{i=1}^n I_{c_i}$.

Thm (Heine-Borel):
Let $\mathcal{G}$ be a collection of open intervals s.t.

$$[a,b] \subseteq \bigcup \mathcal{G}.$$

Then there exists $\{I_1, I_2, \dots, I_n\} \subseteq \mathcal{G}$ s.t. $[a,b] \subseteq \bigcup_{i=1}^n I_i$.

Thm: If $f:[a,b] \to \mathbb{R}$ is continuous, then f is bounded on $[a,b]$.

Pf By Lemma, $\forall c \in [a,b]$, $\exists$ open interval I_c containing c s.t. f is bdd on I_c.

Since $[a,b] \subseteq \bigcup_{c \in [a,b]} I_c$, by Heine-Borel, $\exists \{c_1, \dots, c_n\} \subseteq [a,b]$ s.t. $[a,b] \subseteq \bigcup_{i=1}^{n} I_{c_i}$.

Since f is bdd on $\bigcup_{i=1}^{n} I_{c_i}$, it is bdd on $[a,b]$. $\square$

Thm: If $f:[a,b] \to \mathbb{R}$ is cts, then $\exists\, c, d \in [a,b]$ s.t.

$$f(c) \le f(x) \le f(d), \quad \forall x \in [a,b].$$

Pf: By previous theorem,
$\{f(x) : x \in [a,b]\}$
is a bounded subset of $\mathbb{R}$.
Thus, its supremum exists. Let
$$M = \sup\{f(x) : x \in [a,b]\}.$$

We seek to show that, in fact M is an element of this set, hence is the maximum. That is, we seek $d \in [a,b]$ s.t. $f(d) = M$.

Assume, for the sake of contradiction, that $f(x) < M$ $\forall$ $x \in [a,b]$. Define $g : [a,b] \to \mathbb{R}$ by $g(x) = \dfrac{1}{M - f(x)}$.

Since $h_1(x) = 1$ and $h_2(x) = m - f(x)$ are cts at all $x \in [a,b]$ and $h_2(x) \neq 0$ $\forall x \in [a,b]$, by theorem from last time, g is cts on $[a,b]$.

By prev theorem, g is bounded on $[a,b]$, so $\exists$ $\tilde{m}$ s.t. $\forall x \in [a,b]$,

$$\frac{1}{\underbrace{m-f(x)}_{>0}} < \underbrace{\tilde{m}}_{\text{hence } \tilde{m} > 0}$$

$$\Updownarrow$$

$$\frac{1}{\tilde{m}} < m - f(x)$$

$$\Updownarrow$$

$$f(x) < m - \frac{1}{\tilde{m}}$$

This contradicts that M was the least upper bound of f on $[a,b]$. Thus, $\exists\, d \in [a,b]$ s.t.

$$f(x) \leq f(d), \ \forall\, x \in [a,b].$$

Finally, to see that $\exists\, c \in [a,b]$ s.t. $f(c) \leq f(x)\ \forall\, x \in [a,b]$, note that $-f(x)$ is cts on $[a,b]$. By what we just showed, $\exists\, c \in [a,b]$ s.t.

$$-f(x) \leq -f(c) \quad \forall\, x \in [a,b]$$

$$\Updownarrow$$

$$f(c) \leq f(x) \qquad \square$$

<u>Good news</u>: we now know optimization problems of the

form

cts

$$\min_{x\in[a,b]} f(x)$$

have a solution.

Bad news: Many important functions that we want to optimize are not continuous, but merely lower semicontinuous.

Def: Given $X \subseteq \mathbb{R}$, $f: X \to \overline{\mathbb{R}}$ is
upper semicontinuous at x_0 if
lower semicontinuous

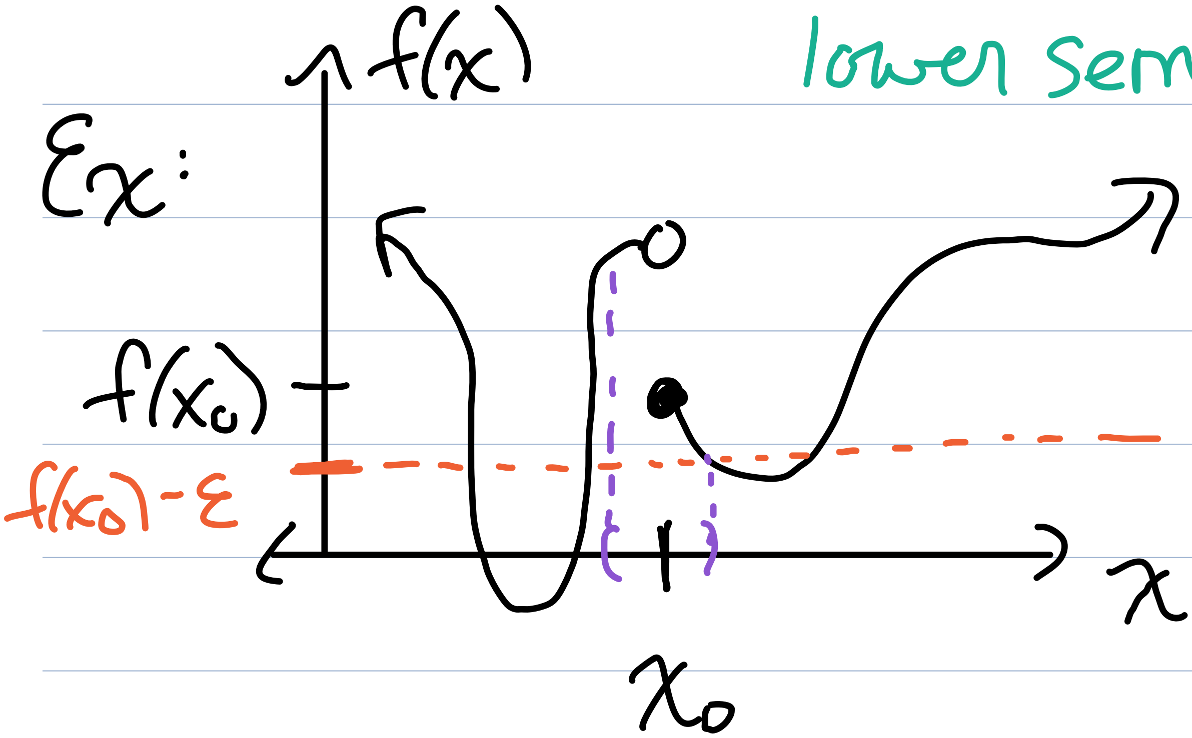
$\forall \varepsilon>0, \exists \delta>0$ s.t. $x \in X$ and $|x-x_0|<\delta$, then

$f(x)-f(x_0)<\varepsilon$

$f(x)-f(x_0)>-\varepsilon \iff f(x)>f(x_0)-\varepsilon$

lower semicontinuous

Ex:



Fact: $f: X \to \overline{\mathbb{R}}$ is lower semicts

$\Updownarrow$

$-f$ is upper semicts

Thm: (HW7) If $f:[a,b] \to \overline{\mathbb{R}}$ is lower semicontinuous, then

$\exists\, c \in [a,b]$ s.t.

$$f(c) \leq f(x), \quad \forall x \in [a,b].$$

f attains its minimum

As before with cts fns, there is also a useful sequential characterization of lower semicty.

20-21 The limsup and liminf of bdd and unbdd sequences

Recall: For $X \subseteq \mathbb{R}$,

$$\sup(X) = \begin{cases} +\infty, & \text{if } X \text{ is unbdd above} \\ \text{supremum of } X & \text{if } X \text{ is bdd above.} \end{cases}$$

Now, for $X \subseteq \overline{\mathbb{R}}$, $X \neq \emptyset$,

$$\sup(X) = \begin{cases} +\infty & \text{if } +\infty \in X \\ -\infty & \text{if } X = \{-\infty\} \\ \sup(X \setminus \{-\infty\}) & \text{if } +\infty \notin X \text{ and } X \neq \{-\infty\} \end{cases}$$

Def: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$

$$\limsup_{n\to\infty} x_n := \lim_{N\to\infty} \overbrace{\sup\{x_n : n > N\}}^{a_N : \mathbb{N} \to \overline{\mathbb{R}}}$$

$$\liminf_{n\to\infty} x_n := \lim_{N\to\infty} \underbrace{\inf\{x_n : n > N\}}_{b_N : \mathbb{N} \to \overline{\mathbb{R}}},$$

For any $x_n : \mathbb{N} \to \overline{\mathbb{R}}$,
these exists.

Lemma: For any $x_n : \mathbb{N} \to \overline{\mathbb{R}}$, a_N is decreasing and b_N is increasing.

Rmk: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$

$$\sup\{-x_n : n > N\} = \begin{cases} +\infty & \text{if } x_n = -\infty \text{ for some } n > N \\ -\infty & \text{if } x_n = +\infty \text{ for all } n > N \\ \sup(\{-x_n : n > N\} \setminus \{-\infty\}) & \text{otherwise} \end{cases}$$

by (*)

$$= \begin{cases} +\infty & \text{if } -x_n = +\infty \text{ for some } n > N \\ -\infty & \text{if } -x_n = -\infty \ \forall\ n > N \\ -\inf(\{x_n : n > N\} \setminus \{+\infty\}) & \text{otherwise} \end{cases}$$

$$= -\inf(\{x_n : n > N\})$$

On HW2, Q5, for $S \subseteq \mathbb{R}$ bdd below
$\sup(-S) = -\inf(S)$.
Furthermore, if $S \subseteq \mathbb{R}$ is not bounded below,
$\sup(-S) = +\infty = -(-\infty) = -\inf(S)$.

So for all $S \subseteq \mathbb{R}$, $\sup(-S) = -\inf(S)$.
$(\star)$

Pf of Lemma:
We will show a_N is decreasing.
If $\exists\, N$ s.t. $x_n = -\infty \;\forall\, n > N$,
then $a_M = -\infty \;\forall\, M \geq N$.

If $a_N \in \mathbb{R}$, then
$\{x_n : n > N\}$ is bounded above
and $\{x_n : n > N+1\} \subseteq \{x_n : n > N\}$,
$a_{N+1} \in \mathbb{R}$ and $a_{N+1} \leq a_N$.

Resume next time...