Lecture 15

Office Hours:

Thur 12-2pm

Midterm 2: Wednesday, May 29th

Textbook? Elementary Analysis, Ross

Thm: If $f:[a,b] \to \mathbb{R}$ is continuous, then f is bounded on $[a,b]$.

Thm: If $f:[a,b] \to \mathbb{R}$ is cts, then $\exists\, c, d \in [a,b]$ s.t.

$$f(c) \le f(x) \le f(d), \quad \forall x \in [a,b].$$

Def: Given $\mathcal{X} \subseteq \mathbb{R}$, $f: \mathcal{X} \to \overline{\mathbb{R}}$ is

$\begin{cases} \text{upper semicontinuous} \\ \text{lower semicontinuous} \end{cases}$ at x_0 if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x \in \mathcal{X}$ and $|x - x_0| < \delta$, then

$\begin{cases} f(x) - f(x_0) < \varepsilon \\ f(x) - f(x_0) > -\varepsilon \iff f(x) > f(x_0) - \varepsilon \end{cases}$

Fact: $f: \mathcal{X} \to \overline{\mathbb{R}}$ is lower semicts

$\Updownarrow$

$-f$ is upper semicts

Thm: (HW7) If $f: [a,b] \to \overline{\mathbb{R}}$ is lower semicontinuous, then $\exists\, c \in [a,b]$ s.t.

$$f(c) \leq f(x), \quad \forall x \in [a,b].$$

f attains its minimum

20-21 The limsup and liminf of bdd and unbdd sequences

Recall: For $X \subseteq \mathbb{R}$,

$$\sup(X) = \begin{cases} +\infty, & \text{if } X \text{ is unbdd above} \\ \text{supremum of } X & \text{if } X \text{ is bdd above.} \end{cases}$$

Now, for $X \subseteq \overline{\mathbb{R}}$, $X \neq \emptyset$,

$$\sup(X) = \begin{cases} +\infty & \text{if } +\infty \in X \\ -\infty & \text{if } X = \{-\infty\} \\ \sup(X \setminus \{-\infty\}) & \text{if } +\infty \notin X \text{ and } X \neq \{-\infty\} \end{cases}$$

$$\inf(X) = \begin{cases} -\infty & \text{if } -\infty \in X \\ +\infty & \text{if } X = \{+\infty\} \\ \inf(X \setminus \{+\infty\}) & \text{otherwise} \end{cases}$$

Lemma: For $X \subseteq Y \subseteq \overline{\mathbb{R}}$,

(i) $\sup(-X) = -\inf(X)$

(ii) $\sup(X) \leq \sup(Y)$

(iii) $\inf(X) \geq \inf(Y)$

Pf: First, we show (i).

$$\sup(-X) = \begin{cases} +\infty & \text{if } +\infty \in -X \\ -\infty & \text{if } -X = \{-\infty\} \\ \sup(-X \setminus \{-\infty\}) & \text{otherwise} \end{cases}$$

$$= \begin{cases} -(-\infty) & \text{if } -\infty \in X \\ -(+\infty) & \text{if } X = \{+\infty\} \\ -\inf(X \setminus \{+\infty\}) & \text{otherwise} \end{cases}$$

$$= -\inf(X).$$

Now, we show (ii).

$$\sup(X) = \begin{cases} +\infty & \text{if } +\infty \in X \\ -\infty & \text{if } X = \{-\infty\} \\ \sup(X\setminus\{-\infty\}) & \text{otherwise} \end{cases}$$

$\to +\infty \in Y$

since $X \neq \{-\infty\}$, so $Y \neq \{-\infty\}$; likewise $X\setminus\{-\infty\} \subseteq Y\setminus\{-\infty\}$

$$\leq \begin{cases} \sup(Y) & \text{if } +\infty \in X \\ \sup(Y) & \text{if } X = \{-\infty\} \\ \sup(Y\setminus\{-\infty\}) & \text{otherwise} \end{cases}$$

$$= \sup(Y)$$

Finally, (iii) follows from (i) and (ii). $\square$

Def: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$

$$\limsup_{n\to\infty} x_n := \lim_{N\to\infty} \overbrace{\sup\{x_n : n > N\}}^{a_N : \mathbb{N} \to \overline{\mathbb{R}}}$$

$$\liminf_{n\to\infty} x_n := \lim_{N\to\infty} \underbrace{\inf\{x_n : n > N\}}_{b_N : \mathbb{N} \to \overline{\mathbb{R}}}$$

For any $x_n: \mathbb{N} \to \overline{\mathbb{R}}$, these exist.

Lemma: For any $x_n: \mathbb{N} \to \overline{\mathbb{R}}$, a_N is decreasing and b_N is increasing. Hence $\lim_{N\to\infty} a_N$ and $\lim_{N\to\infty} b_N$ exist.

Pf of Lemma:
This follows from the fact that $\{x_n : n > N\} \supseteq \{x_n : n > N+1\}$ and the previous lemma.

Ex: $x_n = (-1)^n$

$\limsup_{n\to\infty} x_n = \lim_{N\to\infty}\left(\sup\{x_n : n > N\}\right) = \lim_{N\to\infty} 1 = 1$

$\liminf_{n\to\infty} x_n = -1$

Ex: $x_n = \frac{1}{n}$, $x_n : \mathbb{N} \to \mathbb{R}$

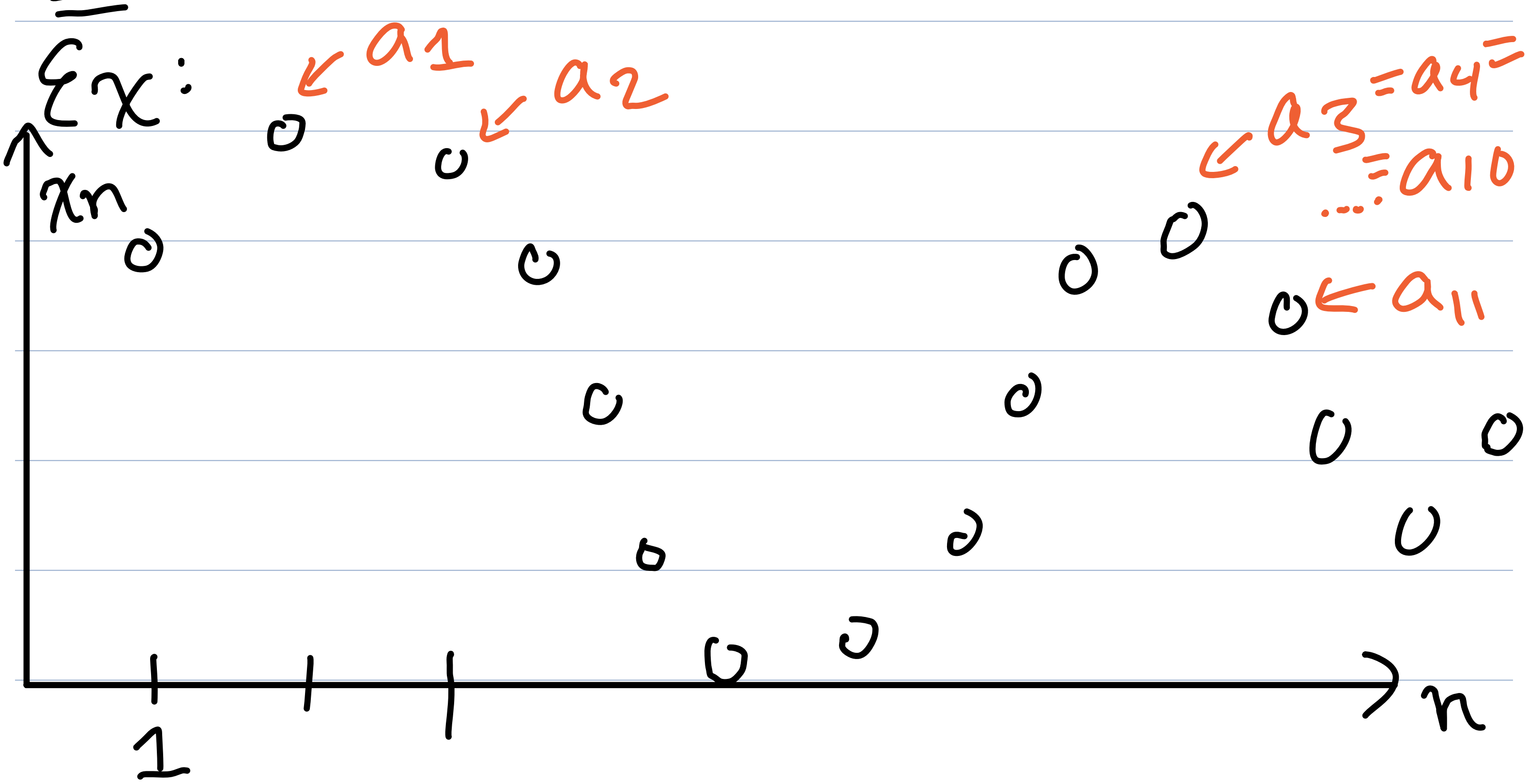
$$\limsup_{n\to\infty} x_n = \lim_{N\to\infty} \sup\{\tfrac{1}{n} : n > N, n \in \mathbb{N}\}$$

$$= \lim_{N\to\infty} \frac{1}{N+1} = 0$$

$$\liminf_{n\to\infty} x_n = \lim_{N\to\infty} \inf\{\tfrac{1}{n} : n > N\}$$

$$= \lim_{N\to\infty} 0 = 0$$

Ex:



Q: Is a_N always a subsequence of x_n?

A: No. Not in previous example.

Also, if $x_n = -\frac{1}{n}$, a_N is $(0, 0, 0, \ldots)$

<u>Thm</u>: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$,

$\lim_{n\to\infty} x_n$ exists $\iff \liminf_{n\to\infty} x_n = \limsup_{n=\infty} x_n$

Furthermore, if either of these equivalent conditions holds,

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n=\infty} x_n.$$

A few facts we will use in proof...

① $b_N = \inf\{x_n : n > N\} \le \sup\{x_n : n > N\}$
$\phantom{b_N = \inf\{x_n : n > N\} \le \sup\{x_n : n > N\}} = a_N$

also true if $\lim r_n$ and $\lim s_n$ exist

② HWQ10: If r_n, s_n converge and $r_n \le s_n \ \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} r_n \le \lim_{n\to\infty} s_n$.

Combining ① and ②

$$\liminf_{n\to\infty} x_n = \lim_{N\to\infty} b_N \le \lim_{N\to\infty} a_N = \limsup_{n\to\infty} x_n$$

We also observe

$$\limsup_{n\to\infty} -x_n = \lim_{N\to\infty} \sup\{-x_n : n > N\}$$

$$= \lim_{N\to\infty} -\inf\{x_n : n > N\}$$

$$= -\lim_{N\to\infty} \inf\{x_n : n > N\}.$$

$$= -\liminf_{n\to\infty} x_n$$

Pf: Suppose $\lim_{n\to\infty} x_n$ exists.

Case 1: $\lim_{n\to\infty} x_n = -\infty$

Fix $M \in \mathbb{R}$. Then $\exists$ N_0 s.t. $n > N_0$ ensures $x_n < M$. Thus

$a_{N_0} = \sup\{x_n : n > N_0\} \le M$.

Since a_N is decreasing, $N \ge N_0$ ensures $a_N \le M$. Hence $\lim_{N\to\infty} a_N = -\infty$.

Thus $\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n = -\infty$, which shows $\lim x_n = \underbrace{\liminf_{n\to\infty} x_n}_{\underline{\lim}\, x_n} = \underbrace{\limsup_{n\to\infty} x_n}_{\overline{\lim}\, x_n}$

Case 2: $\lim_{n\to\infty} x_n = +\infty$

Then $\lim_{n\to\infty} -x_n = -\infty$, by previous part $\limsup -x_n = \liminf -x_n = -\infty$, so $\limsup x_n = \liminf x_n = +\infty$.

Case 3: $\lim_{n\to\infty} x_n = x$, for $x \in \mathbb{R}$.

Fix $\varepsilon > 0$. Then $\exists\, N_0$ s.t. $n > N_0$ ensures $x - \varepsilon < x_n < x + \varepsilon$. Thus

$a_{N_0} = \sup\{x_n : n > N_0\} \le x + \varepsilon$

and $b_{N_0} = \inf\{x_n : n > N_0\} \ge x - \varepsilon$.

Since a_N decreasing, b_N increasing, $\forall\, N \ge N_0$,

$$x - \varepsilon \le b_{N_0} \le b_N \le a_N \le a_{N_0} \le x + \varepsilon$$

Thus $\underbrace{\lim_{N\to\infty} b_N}_{\liminf x_n} = \underbrace{\lim_{N\to\infty} a_N}_{\limsup x_n} = x$

Now, suppose $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Case 1: $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = -\infty$

Fix $M \in \mathbb{R}$. Then $\exists\, N_0$ s.t. $\underbrace{a_{N_0}}_{\sup\{x_n : n > N_0\}} \le M$

Thus $n > N_0$, $x_n \leq M$.
Hence $\lim_{n\to\infty} x_n = -\infty$.

<u>Case 2</u>: $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = +\infty$

By Case 1, $\lim_{n\to\infty} -x_n = -\infty$
$\Rightarrow \lim_{n\to\infty} x_n = +\infty$.

<u>Case 3</u>: $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$, for $x \in \mathbb{R}$.

Fix $\varepsilon > 0$. Since $\lim_{N\to\infty} a_N = \lim_{N\to\infty} b_N = x$,
$\exists$ N_1, N_2 s.t. $N \geq N_1$ ensures
$x - \varepsilon < a_N < x + \varepsilon$ and $N \geq N_2$ ensures
$x - \varepsilon < b_N < x + \varepsilon$. Thus $N = \max\{N_1, N_2\}$,
ensures that for $n > N$
$x - \varepsilon < b_N \leq x_n \leq a_N < x + \varepsilon$.

Thus $\lim_{n\to\infty} x_n = x$. $\square$

We already saw that a_N and b_N are not subsequences of x_n... However, they are closely related... "the set of subsequential limits of x_n"

Thm: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$, let $S = \{s \in \overline{\mathbb{R}} : s \text{ is a limit of a subsequence of } x_n\}$.

Then $\limsup_{n\to\infty} x_n = \max(S)$

$\liminf_{n\to\infty} x_n = \min(S)$.

Ex: If $x_n = \frac{1}{n}$, $S = \{0\}$.

If $x_n = (-1)^n$, $S = \{-1, 1\}$

Prop (HW8):

Consider $x_n : \mathbb{N} \to \mathbb{R}$.

(i) Fix $x \in \mathbb{R}$.

[The set $\{n : |x_n - x| < \varepsilon\}$ is infinite for all $\varepsilon > 0$.]

$\Updownarrow$

x is a subsequential limit of x_n

(ii) x_n is unbdd above

$\Updownarrow$

$+\infty$ is a subsequential limit of x_n

(iii) x_n is unbdd below

$\Updownarrow$

$-\infty$ is a subsequential limit of x_n.