Lecture 16

Office Hours:

Tues 2:30-3:30pm, Thurs 1-2pm

Final Exam: Thurs, June 13, 12-2pm

Recall:

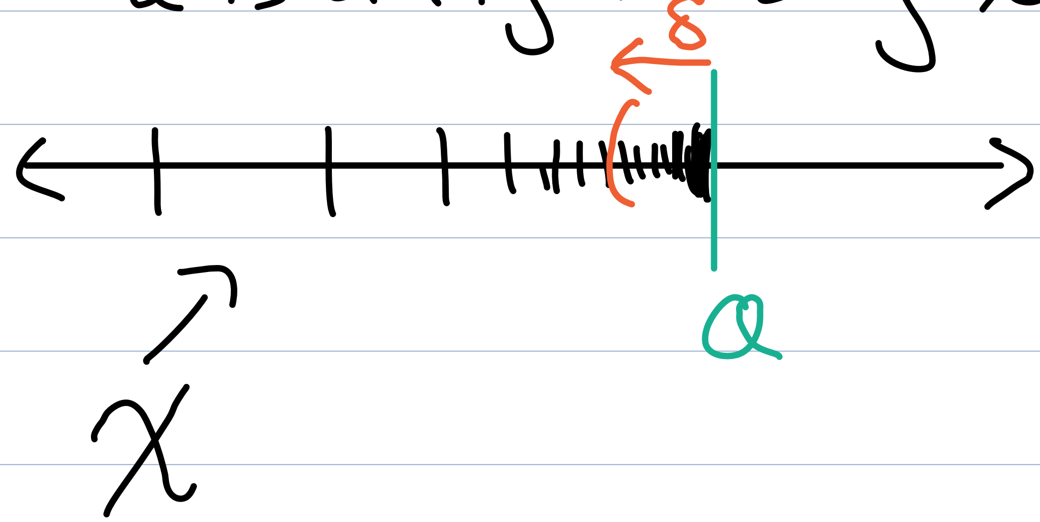
Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$ is

a $\begin{cases} \cancel{\text{right}}\ \text{left acc point of } X \\ \cancel{\text{left}}\ \text{right acc point of } X \end{cases}$

if $\forall \delta > 0$, $\exists x \in X$ s.t.

$\begin{cases} 0 < a - x < \delta \\ 0 < x - a < \delta \end{cases}$

a is a ~~right~~ left acc of X



Lemma: a is a ~~right~~ left (resp. ~~left~~ right) acc point of $X \subseteq \mathbb{R}$

$$\Updownarrow$$

$\exists\, x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \nearrow a$

$x_n \searrow a$

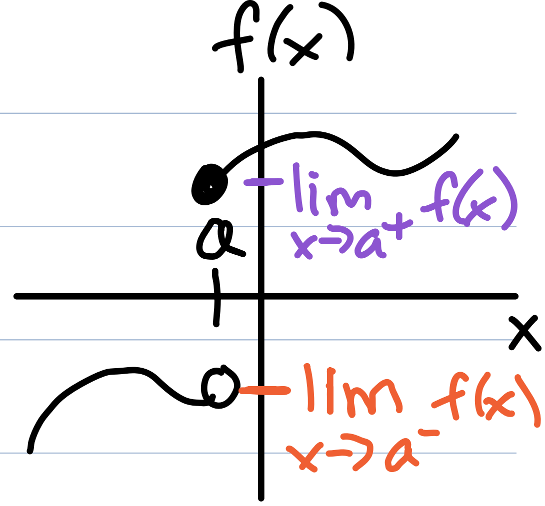
Def: Given $X \subseteq \mathbb{R}$, $f : X \to \overline{\mathbb{R}}$, a a $\begin{cases} \cancel{\text{right}}\ \text{left acc pt} \\ \cancel{\text{left}}\ \text{right acc pt} \end{cases}$ of X, $L \in \overline{\mathbb{R}}$, the limit of $f(x)$ as x approaches a $\begin{cases} \text{from the left} \\ \text{from the right} \end{cases}$ is L if

$\forall\, x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $\begin{cases} x_n \nearrow a \\ x_n \searrow a \end{cases}$, $\lim\limits_{n\to\infty} f(x_n) = L$.

We denote this as

$$\lim_{x\to a^-} f(x) = L$$

$$\lim_{x\to a^+} f(x) = L$$



<u>Def</u>: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$

$$\limsup_{n\to\infty} x_n := \lim_{N\to\infty} \overbrace{\sup\{x_n : n > N\}}^{a_N : \mathbb{N} \to \overline{\mathbb{R}}}$$

$$\liminf_{n\to\infty} x_n := \lim_{N\to\infty} \underbrace{\inf\{x_n : n > N\}}_{b_N : \mathbb{N} \to \overline{\mathbb{R}}},$$

Lemma: For any $x_n : \mathbb{N} \to \overline{\mathbb{R}}$, a_N is decreasing and b_N is increasing. Hence $\lim_{N\to\infty} a_N$ and $\lim_{N\to\infty} b_N$ exist.

Thm: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$,

$$\lim_{n\to\infty} x_n \text{ exists} \iff \liminf_{n\to\infty} x_n = \limsup_{n=\infty} x_n$$

Furthermore, if either of these equivalent conditions holds,

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n=\infty} x_n .$$

Fact: $\liminf_{n\to\infty} -x_n = -\limsup_{n\to\infty} x_n$.

$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$

Thm (HW7): Consider $x_n : \mathbb{N} \to \overline{\mathbb{R}}$.

(i) Fix $x \in \mathbb{R}$.

x is a subsequential limit

$\Updownarrow$

$\forall \varepsilon > 0, |\{n : |x_n - x| < \varepsilon\}| = +\infty$

(ii) $+\infty$ is a subsequential limit

$\Updownarrow$ {x_n: n>N} is unbdd above for all N

~~x_n is unbounded above~~

(iii) $-\infty$ is a subsequential limit

$\Updownarrow$ {x_n: n>N} is unbdd below for all N

~~x_n is unbounded below~~

Thm: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$, let $S = \{ s \in \overline{\mathbb{R}} : s \text{ is a limit of a subsequence of } x_n \}$.

Then $\limsup_{n\to\infty} x_n = \max(S)$

$\liminf_{n\to\infty} x_n = \min(S)$.

Pf:

Step 1: We will show $\limsup_{n\to\infty} x_n \in S$.

$\boxed{\text{Case 1}}$: $\limsup_{n\to\infty} x_n = -\infty$

Then $\limsup x_n \geq \liminf x_n \Rightarrow \liminf x_n = -\infty$.

Thus, by prev thm, $\lim_{n\to\infty} x_n = -\infty$, so $S = \{-\infty\}$. Thus $\limsup_{n\to\infty} x_n \in S$.

$\boxed{\text{Case 2}}$: $\limsup_{n\to\infty} x_n = +\infty$.

That is, $\lim_{N\to\infty} a_N = +\infty$. Fix arbitrary $M \in \mathbb{R}$. Then $\exists N_0$ s.t. $N \geq N_0$ ensures $a_N > M$.

Since $a_N = \sup\{x_n : n > N\}$, M is not an upper bound for $\{x_n : n > N\}$, so $\exists n_0 > N$ s.t. $x_{n_0} > M$. Thus ~~x_n is unbounded above, hence~~ $\limsup x_n = +\infty \in S$.

$\boxed{\text{Case 3}}$: $\limsup_{n\to\infty} x_n = t \in \mathbb{R}$...

that is $\lim_{N\to\infty} a_N = t$. Fix $\varepsilon > 0$.

By defn of convergence, $\exists N_0$ s.t. $N \geq N_0$ ensures

$|a_N - t| < \varepsilon \implies$

$\sup\{x_n : n > N\} = a_N < t + \varepsilon.$

Thus, $n > N_0$ ensures $x_n < t + \varepsilon$.

Suppose, for the sake of contradiction, that $|\{n : t - \varepsilon < x_n < t + \varepsilon\}| < +\infty$.

Thus, $\exists$ $N_1 > N_0$ s.t. $x_n \leq t - \varepsilon$ for all $n > N_1$.

Thus, for all $N \geq N_1$,

$a_N \leq a_{N_1} = \sup\{x_n : n > N_1\} \leq t - \varepsilon.$

This contradicts that $\lim_{N \to \infty} a_N = t$.

Thus, $|\{n : t - \varepsilon < x_n < t + \varepsilon\}| = +\infty$,

so $\limsup x_n = t \in S$.

Step 2:

Note that $\liminf_{n\to\infty} x_n = -\limsup_{n\to\infty} -x_n$.

Note that if S is the set of subsequential limits of x_n, then $-S$ is the set of subsequential limits of $-x_n$.

Thus, since Step 1 showed $\limsup_{n\to\infty} -x_n \in -S$, we have $\liminf_{n\to\infty} x_n = -\limsup_{n\to\infty} -x_n \in S$.

Step 3:

We will now show $\limsup x_n$ and $\liminf x_n$ are the largest and smallest subsequential limits.

Fix $t \in S$. There exists $x_{n_k} \to t$.

Since $n_k \geq k$, for all $N \in \mathbb{N}$
$\{x_{n_k} : k > N\} \subseteq \{x_n : n > N\}$.

Thus

$$b_N = \inf\{x_n : n > N\} \leq \inf\{x_{n_k} : k > N\}$$

$$\sup\{x_{n_k} : k > N\} \leq \sup\{x_n : n > N\} = a_N$$

Sending $N \to \infty$,

$$\liminf_{n\to\infty} x_n = \lim_{N\to\infty} b_N \leq \liminf_{k\to\infty} x_{n_k} = t$$

$$\ldots = \limsup_{k\to\infty} x_{n_k} \leq \lim_{N\to\infty} a_N = \limsup_{n\to\infty} x_n.$$

$\square$

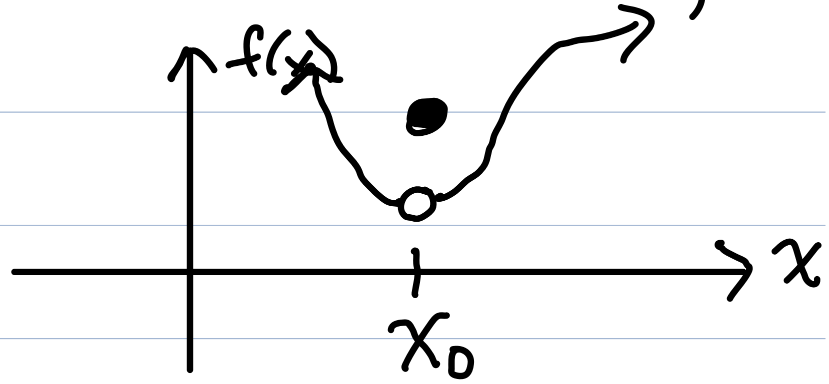
Application of liminf and limsup: sequential characterization of usc/lsc.

Thm: Given $X \subseteq \mathbb{R}$, $f: X \to \overline{\mathbb{R}}$ is $\left\{\begin{array}{l}\text{upper semicts} \\ \text{lower semicts}\end{array}\right.$ at $x_0 \in X$

$$\Updownarrow$$

$$\boxed{\forall\, x_n : \mathbb{N} \to X \text{ s.t. } x_n \to x_0, \left\{\begin{array}{l}\limsup f(x_n) \le f(x_0) \\ \liminf f(x_n) \ge f(x_0)\end{array}\right.} \quad (*)$$

Pf: First assume f is upper semicts at x_0, that is, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $x \in X$ and $|x - x_0| < \delta$, $f(x) < f(x_0) + \varepsilon$.



Fix $x_n : \mathbb{N} \to X$ s.t. $x_n \to x_0$.
Let $a_N = \sup\{f(x_n) : n > N\}$.
We seek to show $\lim_{N\to\infty} a_N \le f(x_0)$.

Fix $\varepsilon > 0$ arbitrary. It suffices to show that $\exists N_0$ s.t. $N \ge N_0$ ensures $a_N \le f(x_0) + \varepsilon$. Then we will have $\lim_{N\to\infty} a_N \le f(x_0) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{N\to\infty} a_N \le f(x_0)$.

Choose $\delta > 0$ as in the definition of upper semicontinuity. Since $x_n \to x_0$, $\exists N_0$ s.t. $n \ge N_0$ ensures $|x_n - x_0| < \delta$; hence $f(x_n) < f(x_0) + \varepsilon$. This shows

$a_{N_0} \leq f(x_0) + \varepsilon$.

Since a_N is decreasing, $\forall N \geq N_0$,

$a_N \leq a_{N_0} \leq f(x_0) + \varepsilon$.

Thus $(\ast)$ holds.

Assume $(\ast)$ holds. We seek to show f is upper semicts. Fix $\varepsilon > 0$ arb. Assume, for the sake of contradiction that, $\forall \delta > 0$, $\exists x \in X$ with $|x - x_0| < \delta$ but $f(x) \geq f(x_0) + \varepsilon$. Thus, there exists $x_n : \mathbb{N} \to X$ s.t. $x_n \to x_0$ and $f(x_n) \geq f(x_0) + \varepsilon$, $\forall n \in \mathbb{N}$. Hence $\sup\{f(x_n) : n > N\} \geq f(x_0) + \varepsilon$. Thus $\limsup_{n \to \infty} f(x_n) \geq f(x_0) + \varepsilon > f(x_0)$.

This is a contradiction. Thus f is upper semicontinuous.

Finally, the corresponding characterization of lsc follows from the fact that f is lsc $\Leftrightarrow$ $-f$ is usc. $\square$

One last important property of continuous functions:

<u>Thm</u>: (Intermediate Value Thm) Given interval $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ s.t. f is cts at x for all $x \in I$, then, for any $a, b \in I$,

if y is between $f(a)$ and $f(b)$,

$f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$

then $\exists$ x between a and b

$a \leq x \leq b$ or $b \leq x \leq a$

s.t. $f(x) = y$.