Lecture 17

Midterm 2 solutions posted

Office Hours: Thurs 1-2pm

Final Exam: Thurs, June 13, 12-2pm

Recall:

Thm: Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$, let $S = \{s \in \overline{\mathbb{R}} : s \text{ is a limit of a subsequence of } x_n\}$.

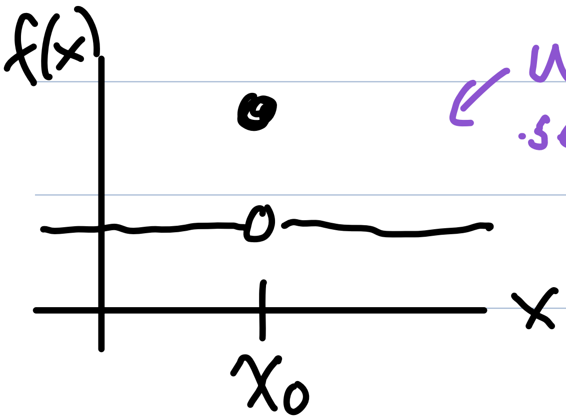
Then $\limsup_{n\to\infty} x_n = \max(S)$

$\liminf_{n\to\infty} x_n = \min(S)$.

Thm: Given $X \subseteq \mathbb{R}$, $f: X \to \overline{\mathbb{R}}$ is
$\begin{cases} \text{upper semicts} \\ \text{lower semicts} \end{cases}$ at $x_0 \in X$

$\Updownarrow$

$\forall\, x_n : \mathbb{N} \to X$ s.t. $x_n \to x_0$, $\begin{cases} \limsup f(x_n) \le f(x_0) \\ \liminf f(x_n) \ge f(x_0) \end{cases}$



upper semicts

One last important property of continuous functions:

Thm: (Intermediate Value Thm)
Given interval $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ s.t. f is cts at x for all $x \in I$, then, for any $a, b \in I$,

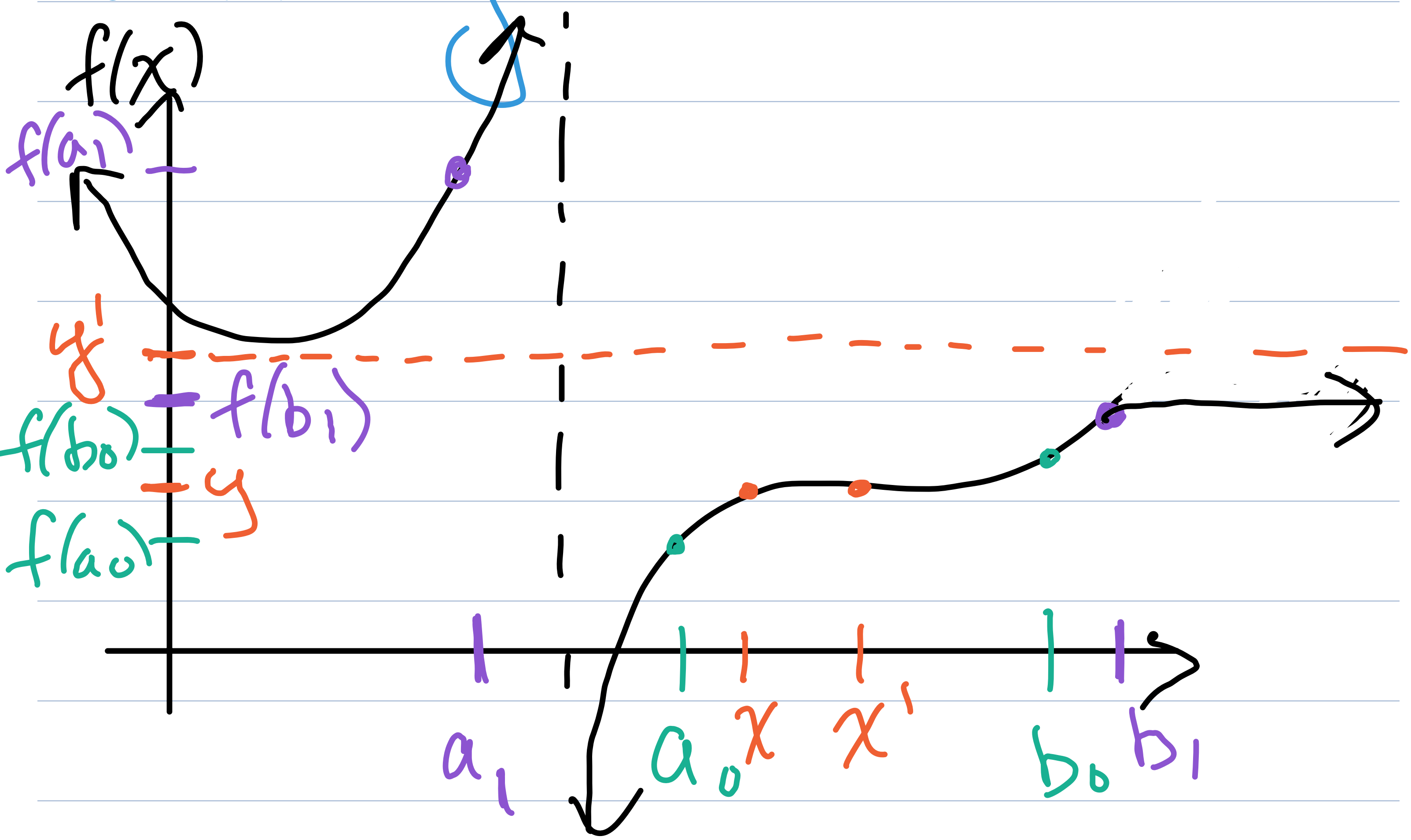
if y is between $f(a)$ and $f(b)$,

$f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$

then $\exists$ x between a and b

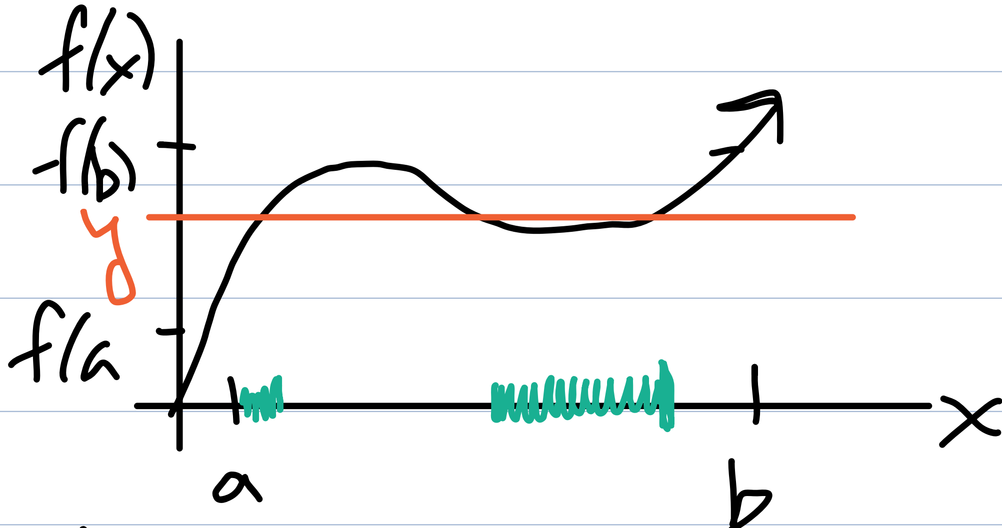
$a \leq x \leq b$ or $b \leq x \leq a$

s.t. $f(x) = y$.



Pf: Fix $a, b \in I$.
WLOG $a \leq b$. Suppose y is between $f(a)$ and $f(b)$.

Case 1: $f(a) \leq y \leq f(b)$
We must find $x \in [a,b]$ s.t. $f(x) = y$



Let $S = \{x \in [a,b] : f(x) \leq y\}$.
Since $a \in S$, $S \neq \emptyset$. Since S is bdd above, its supremum exists.
Let $x_0 = \sup(S)$.

First, we will show $f(x_0) \leq y$.

For any $n \in \mathbb{N}$, $x_0 - \frac{1}{n}$ is not an upper bound of S so $\exists\, x_n \in S$ $x_0 \geq x_n > x_0 - \frac{1}{n}$. Thus $x_n \to x_0$.

Furthermore, $x_n \in S$ ensures $f(x_n) \leq y$ $\forall n \in \mathbb{N}$. Since f is cts at x_0,

$\lim_{x \to x_0} f(x) = f(x_0)$.

$$y \geq \lim_{n \to \infty} f(x_n) = \lim_{x \to x_0} f(x) = f(x_0).$$

It remains to show $f(x_0) \geq y$.

If $x_0 = b$, $f(x_0) = f(b) \geq y$. Now, suppose $x_0 < b$. Define

$$t_n = \min\{b, x_0 + \tfrac{1}{n}\}.$$

By definition,

$$x_0 < t_n \leq x_0 + \frac{1}{n}.$$

Thus $t_n \to x_0$.

Since $t_n > x_0$, $t_n \notin S$,
but since $t_n \in [a, b]$, we
must have $f(t_n) > y$.

$y \leq \lim_{n\to\infty} f(t_n) = \lim_{x \to x_0} f(x) = f(x_0)$.

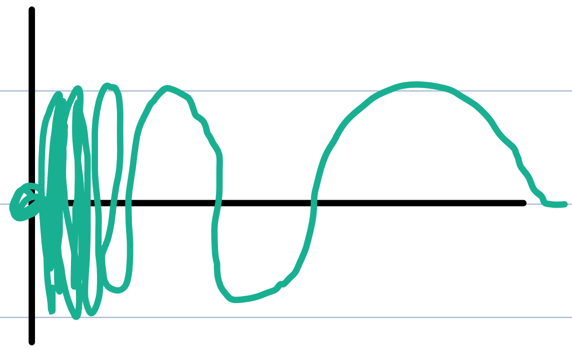
This shows $f(x_0) = y$.

<u>Case 2</u>: $f(b) \leq y \leq f(a)$
Since $-f$ is continuous on
I, by case 1, $\exists$ $x_0 \in [a,b]$
s.t. $-f(x_0) = -y$. This
gives $f(x_0) = y$. $\square$

Moral: cts fns always attain
"intermediate values."

Interestingly, the intermediate value property does not characterize continuous functions.

Ex: $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$



Claim: f is discontinuous at 0, that is we do not have $\lim_{x \to 0} f(x) = 0$.

Let $x_n = \left(\frac{n\pi}{2}\right)^{-1}$. Then $x_n \to 0$.

However,
$f(x_n) = \sin\left(\frac{n\pi}{2}\right) \not\to 0$.
Thus f is discontinuous at 0.

<u>Claim</u>: f is continuous at all $x \in \mathbb{R} \setminus \{0\}$.

<u>Claim</u>: f satisfies the intermediate value property, that is, for all $a \le b$ [and y between $f(a)$ and $f(b)$], there exists $x \in [a,b]$ s.t. $f(x) = y$.

In fact, the only problem with this example is that there is no open interval around zero on which f is either increasing or decreasing.

Cor: Given an interval $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ cts at all $x \in I$, $f(I)$ is an interval.

Pf: If $\inf(f(I)) = \sup(f(I))$, ~~then~~ then $f(I)$ ~~is~~ is a single point, hence an interval.

Suppose $\inf(f(I)) < \sup(f(I))$ and fix y s.t. $\inf(f(I)) < y < \sup(f(I))$. Then there exist $y_0, y_1 \in f(I)$ $x_0, x_1 \in I$

$f(x_0) = y_0 < y < y_1 = f(x_1)$.

By IVT, there exists $x \in I$ s.t. $f(x) = y$.

Thus, $f(I)$ is an interval.

Thm: Given an interval $I \subseteq \mathbb{R}$, suppose $f: I \to \mathbb{R}$ is strictly increasing and $f(I)$ is an interval. Then f is continuous at all $x \in I$.

$x < y \Rightarrow f(x) < f(y)$

Pf: Fix $x_0 \in I$.

Case 1: $\inf(I) < x_0 < \sup(I)$.

Then, since f is strictly increasing,

$$\inf(f(I)) < f(x_0) < \sup(f(I)).$$

Fix $\varepsilon > 0$. Let
$\varepsilon_0 = \min\{\varepsilon, |f(x_0) - \inf(f(I))|, |f(x_0) - \sup f(I)|\}$
Note $\varepsilon_0 > 0$.

Then

$$\inf(f(I)) < f(x_0) - \frac{\varepsilon_0}{2}$$
$$< f(x_0)$$
$$< f(x_0) + \frac{\varepsilon_0}{2}$$
$$< \sup(f(I))$$

Thus $f(x_0) - \frac{\varepsilon_0}{2} \in f(I)$ and $\exists\, x_1 \in I$, s.t. $f(x_1) = f(x_0) - \frac{\varepsilon_0}{2}$.
Similarly, $\exists\, x_2 \in I$ s.t.
$f(x_2) = f(x_0) + \frac{\varepsilon_0}{2}$.

Since f is strictly increasing,

$$x_1 < x_0 < x_2.$$

Let $\delta = \min\{|x_0 - x_1|, |x_0 - x_2|\}$.
Then if $|x - x_0| < \delta$, we have

$x_1 < x < x_2$, so

$$f(x_0) - \frac{\varepsilon_0}{2} = f(x_1) < f(x) < f(x_2) = f(x_0) + \frac{\varepsilon_0}{2}$$

Thus, $|f(x) - f(x_0)| < \frac{\varepsilon_0}{2} < \varepsilon$.

<u>Case 2</u>: exerise :)