Recall:

**Thm:** Given $x_n : \mathbb{N} \to \overline{\mathbb{R}}$, let

$$S = \{ s \in \overline{\mathbb{R}} : s \text{ is a limit of a subsequence of } x_n \}.$$

Then $\limsup_{n \to \infty} x_n = \max(S)$ and $\liminf_{n \to \infty} x_n = \min(S)$. 
**Thm:** Given $X \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$ is upper semicontinuous at $x_0 \in X$.

Given $X \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$ is lower semicontinuous.

For all $x_0 \in X$ and any sequence $(x_n)_{n=1}^{\infty}$ in $X$ that converges to $x_0$,

\[ \limsup_{n \to \infty} f(x_n) \leq f(x_0), \]

\[ \liminf_{n \to \infty} f(x_n) \geq f(x_0). \]

One last important property of continuous functions:

**Thm:** (Intermediate Value Thm)

Given interval $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ is continuous at $x$ for all $x \in I$.

Then, for any $a, b \in I$,
if \( y \) is between \( f(a) \) and \( f(b) \),
\[
\frac{f(a)}{f(a)} \leq y \leq f(b) \quad \text{or} \quad f(b) \leq y \leq f(a)
\]
then \( \exists x \) between \( a \) and \( b \),
\[
a \leq x \leq b \lor b \leq x \leq a
\]
s.t. \( f(x) = y \).
Proof: Fix $a, b \in I$. WLOG $a \leq b$. Suppose $y$ is between $f(a)$ and $f(b)$.

Case 1: $f(a) \leq y \leq f(b)$

We must find $x \in [a, b]$ s.t. $f(x) = y$.

Let $S = \{ x \in [a, b] : f(x) \leq y \}$.

Since $a \in S$, $S \neq \emptyset$. Since $S$ is bounded above, its supremum exists.

Let $x_0 = \sup(S)$.
First, we will show \( f(x_0) \leq y \).
For any \( n \in \mathbb{N} \), \( x_0 - \frac{1}{n} \) is not an upper bound of \( S \) so \( \exists x_n \in S \)
\( x_0 = x_n > x_0 - \frac{1}{n} \). Thus \( x_n \rightarrow x_0 \).
Furthermore, \( x_n \in S \) ensures \( f(x_n) \leq y \)
\( \forall n \in \mathbb{N} \). Since \( f \) is continuous at \( x_0 \),
\[
\lim_{x \rightarrow x_0} f(x) = f(x_0).
\]
\[
y \geq \lim_{n \rightarrow \infty} f(x_n) = \lim_{x \rightarrow x_0} f(x) = f(x_0).
\]
It remains to show \( f(x_0) = y \).
If \( x_0 = b \), \( f(x_0) = f(b) \geq y \). Now,
suppose \( x_0 < b \). Define
\[
t_n = \min \{ b, x_0 + \frac{1}{n} \}.
\]
By definition,
\[
x_0 < t_n \leq x_0 + \frac{1}{n}.
\]
Thus \( t_n \rightarrow x_0 \).
Since $t_n > x_0$, $t_n \notin S$, but since $t_n \in [a, b]$, we must have $f(t_n) > y$. 

\[
y = \lim_{n \to \infty} f(t_n) = \lim_{x \to x_0} f(x) = f(x_0).
\]

This shows $f(x_0) = y$.

**Case 2**: $f(b) \leq y \leq f(a)$

Since $-f$ is continuous on $I$, by case 1, $\exists x_0 \in [a, b]$ s.t. $-f(x_0) = -y$. This gives $f(x_0) = y$. 

\[\square\]

Moral: cts fns always attain "intermediate values!"
Interestingly, the intermediate value property does not characterize continuous functions.

\[ f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases} \]

Claim: \( f \) is discontinuous at 0, that is we do not have \( \lim_{x \to 0} f(x) = 0 \).

Let \( x_n = \left( \frac{n\pi}{2} \right)^{-1} \). Then \( x_n \to 0 \).
However, \( f(x_n) = \sin \left( \frac{m\pi}{2} \right) \to 0 \).
Thus \( f \) is discontinuous at 0.

\textbf{Claim:} \( f \) is continuous at all \( x \in \mathbb{R} \setminus \{0\} \).

\textbf{Claim:} \( f \) satisfies the intermediate value property, that is, for all \( a \leq b \) and \( y \) between \( f(a) \) and \( f(b) \), there exists \( x \in [a, b] \) s.t. \( f(x) = y \).

In fact, the only problem with this example is that there
There is no open interval around zero on which $f$ is either increasing or decreasing.

**Cor:** Given an interval $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ cts at all $x \in I$, $f(I)$ is an interval.

**Proof:** If $\inf(f(I)) = \sup(f(I))$, then $f(I)$ is a single point, hence an interval.

Suppose $\inf(f(I)) < \sup(f(I))$ and fix $y$ s.t. $\inf(f(I)) < y < \sup(f(I))$. Then there exist $y_0, y_1 \in f(I)$, $x_0, x_1 \in I$. 
\[ f(x_0) = y_0 < y < y_1 = f(x_1). \]

By IVT, there exists \( x \in I \) s.t. \( f(x) = y \).

Thus, \( f(I) \) is an interval.

\[ \text{Thm: Given an interval } I \subseteq \mathbb{R}, \text{ suppose } f: I \to \mathbb{R} \text{ is strictly increasing and } f(I) \text{ is an interval. Then } f \text{ is continuous at all } x \in I. \]

\[ x < y \implies f(x) < f(y) \]
\textbf{Prf:} Fix }x_0 \in I).

\underline{Case 1: } \inf(I) < x_0 < \sup(I).

Then, since \( f \) is strictly increasing,

\[ \inf(f(I)) < f(x_0) < \sup(f(I)). \]

Fix \( \varepsilon > 0 \). Let

\[ \varepsilon_0 = \min \{ \varepsilon, |f(x_0) - \inf(f(I))|, |f(x_0) - \sup(f(I))| \} \]

Note \( \varepsilon_0 > 0 \).
Then
\[ \inf_{f(I)} \leq f(x_0) - \frac{\epsilon_0}{2} < f(x_0) \]
\[ < f(x_0) + \frac{\epsilon_0}{2} < \sup_{f(I)} \]

Thus \( f(x_0) - \frac{\epsilon_0}{2} \in f(I) \) and \( \exists \ x_1 \in I \), s.t. \( f(x_1) = f(x_0) - \frac{\epsilon_0}{2} \).

Similarly, \( \exists \ x_2 \in I \) s.t.
\( f(x_2) = f(x_0) + \frac{\epsilon_0}{2} \).

Since \( f \) is strictly increasing,
\[ x_1 < x_0 < x_2. \]

Let \( \delta = \min \{ |x_0 - x_1|, |x_0 - x_2| \} \).
Then if \( |x - x_0| < \delta \), we have
\[ x_1 < x < x_2 \], so

\[ f(x_0) - \frac{\varepsilon_0}{2} < f(x) < f(x_2) = f(x_0) + \frac{\varepsilon_0}{2} \]

Thus, \[ |f(x) - f(x_0)| < \frac{\varepsilon_0}{2} < \varepsilon \].

Case 2: exercise.