Lecture 1

office hours and makeup lecture poll

CS 117, S24

Johnsonbaugh and Pfaffenberger (J & P)

Ch. 1: Sets and Functions

(See also Tao, Analysis I, for a detailed, set-theoretic presentation)

"A set is any collection of objects"

Ex: $\{1,3,4\}$, $\{x : x \in \mathbb{R}\ x^2 - 1 = 0\}$
$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$

Def: Given a set X, let 2^X denote the collection of all subsets of X.

Def: Given $\mathcal{A} \subseteq 2^X$, let

$$\cup \mathcal{A} = \{x : x \in A \text{ for some } A \in \mathcal{A}\} = \bigcup_{A \in \mathcal{A}} A$$

Def: Given sets X and Y a **function** from X into Y denoted $f: X \to Y$ is a subset $f \subseteq X \times Y$ satisfying

(i) $(x,y) \in f,\ (x,y') \in f \Rightarrow y = y'$

(ii) $x \in X \Rightarrow (x,y) \in f$ for some $y \in Y$

- X is the domain, Y is the codomain
- If $(x,y) \in f$, write $f(x) = y$.
- Given $A \subseteq X$, $f(A) := \{f(x) : x \in A\}$ is the image of A under f
- $f(X)$ is the range of f
- Given $B \subseteq Y$, $f^{-1}(B) := \{x : f(x) \in B\}$ is the preimage of B under f

Thm: Given $f: X \to Y$, $\mathcal{A} \subseteq 2^X$, $\mathcal{C} \subseteq 2^Y$

(i) $f(\bigcup \mathcal{A}) = \bigcup \{f(A) : A \in \mathcal{A}\}$

(ii) $f^{-1}(\bigcup \mathcal{C}) = \bigcup \{f^{-1}(C) : C \in \mathcal{C}\}$

(iii) $f^{-1}(\bigcap \mathcal{C}) = \bigcap \{f^{-1}(C) : C \in \mathcal{C}\}$

(iv) $C \subseteq Y$, $f^{-1}(C^c) = (f^{-1}(C))^c$

Pf: iPad deleted what I wrote; see textbook

Rmk: In general,

(a) $f(\cap \mathcal{A}) \neq \cap\{f(A) : A \in \mathcal{A}\}$

(b) $f(A^c) \neq (f(A))^c$, for $A \subseteq X$

Consider

$X = Y = \{1,2,3\}$

$f(x) = 1 \quad \forall x \in X$

$\mathcal{A} = \{\{1\},\{2\}\}$ → this shows (a) above

$A = \{3\}$ → this shows (b) above

Ch2: The Real Number System

Def: A <u>binary operation</u> on a set X is is a function from $X \times X \to X$.

Def: A <u>field</u> F is a set with

$\boxed{A1}$ a binary operation called addition

$\boxed{A2}$ $(x+y)+z = x+(y+z)$ $\forall x,y,z \in F$

$\boxed{A3}$ $x+y=y+x$ $\forall x,y \in F$

$\boxed{A4}$ $\exists\, 0 \in F$ s.t. $x+0=x$, $\forall x \in F$

$\boxed{A5}$ $\forall x \in F$, $\exists!\, y \in F$ s.t. $x+y=0$

we will denote as $-x$

$\boxed{M1}$ a binary operation called mult.

$\boxed{M2}$ $(xy)z = x(yz)$ $\forall x,y,z \in F$

$\boxed{M3}$ $xy = yx$

$\boxed{M4}$ $\exists\, 1 \in F \setminus \{0\}$, $x1 = x$ $\forall x \in F$

$\boxed{M5}$ $\forall x \in F \setminus \{0\}$, $\exists!\, y \in F$ s.t. $xy = 1$.

we will denote as x^{-1} or $\frac{1}{x}$

$\boxed{DL}$ $x(y+z) = xy + xz$, $\forall x,y,z \in F$

Def: An ordered field is a field satisfying

$\boxed{0}$ There is a subset P of F called the positive numbers satisfying

(i) $x, y \in P \Rightarrow x+y \in P$ and $xy \in P$

(ii) $\forall x \in F$ exactly one of the following holds:

$x \in P$, $x = 0$, $-x \in P$

Given $x, y \in F$

- x is negative if $-x \in P$
- $x > y$ iff $x - y \in P$
- $x \geq y$ if either $x > y$ or $y = x$

Ex: $\mathbb{R}$, $\mathbb{Q}$

Def: Given an ordered field F and a nonempty set $X \subseteq F$

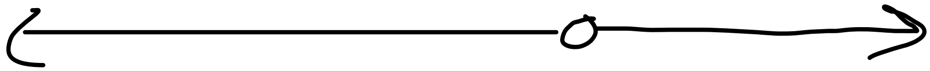
- $a \in F$ is an upper bound of X if $x \leq a \ \forall x \in X$
- X is bounded above if an upper bound exists.

- $a \in F$ is the supremum of X if
  (i) a is an upper bound for X
  (ii) for any upper bound a' of X, $a \leq a'$.

Def: The real numbers $\mathbb{R}$ is the ordered field s.t.
$\forall X \subseteq \mathbb{R}, X \neq \emptyset$, X bounded above the supremum of X exists.

Thm: $\mathbb{R}$ exists.

Rmk: "$\mathbb{R}$ has no holes"



<u>Def</u>: The natural numbers $\mathbb{N}$ is the <u>smallest</u> subset of $\mathbb{R}$ having the properties that $\boxed{\text{(i)}}$ $1 \in \mathbb{N}$ and $\boxed{\text{(ii)}}$ $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$

▷ "smallest" is in the sense of set inclusion

<u>Thm</u>:

(a) If $\mathcal{A} \subseteq 2^{\mathbb{R}}$ is the collection of $A \subseteq \mathbb{R}$ s.t. (i) and (ii) hold, then $\bigcap \mathcal{A}$ satisfies (i) and (ii)

(b) $\mathbb{N} = \bigcap \mathcal{A}$. In particular, $\mathbb{N}$ exists.

Pf:

(a) Since A satisfies (i) $\forall A \in \mathcal{A}$, $1 \in \bigcap \mathcal{A}$. If $n \in \bigcap \mathcal{A}$, then $n \in A$ $\forall A \in \mathcal{A}$, so $n+1 \in A$ $\forall A \in \mathcal{A}$. Hence $n+1 \in \bigcap \mathcal{A}$.

(b) :)