Johnsonbaugh and Pfaffenberger (J & P)
Ch. 1: Sets and Functions
(See also Tao, Analysis I, for a detailed set-theoretic presentation)

"A set is any collection of objects"

\[ \{1, 3, 4\}, \{x \in \mathbb{R} \mid x^2 - 1 = 0\} \]
\[ \mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R} \]

Def: Given a set \( X \), let \( 2^X \) denote the collection of all subsets of \( X \).

Def: Given \( A \subseteq 2^X \), let

\[ \bigcup A = \{ x : x \in A \text{ for some } A \subseteq 2^X \} = \bigcup_{A \subseteq 2^X} A \]
Def: Given sets $X$ and $Y$ a function from $X$ into $Y$ denoted $f: X \rightarrow Y$ is a subset $f \subseteq X \times Y$ satisfying
(i) $(x, y) \in f, (x, y') \in f \implies y = y'$
(ii) $x \in X \implies (x, y) \in f$ for some $y \in Y$

- $X$ is the domain, $Y$ is the codomain
- If $(x, y) \in f$, write $f(x) = y$.
- Given $A \subseteq X$, $f(A) := \{f(x) : x \in A\}$ is the image of $A$ under $f$
- $f(X)$ is the range of $f$
- Given $B \subseteq Y$, $f^{-1}(B) = \{x : f(x) \in B\}$ is the preimage of $B$ under $f$

Thm: Given $f: X \rightarrow Y$, $cA \subseteq 2^X, c \subseteq 2^Y$
(i) $f(UcA) = U\{f(A) : A \in cA\}$
(ii) $f^{-1}(UC) = U\{f^{-1}(C) : C \in c\}$
(iii) $f^{-1}(\cap c) = \cap f^{-1}(C) : C \in c$
(iv) $C \subseteq Y$, $f^{-1}(C^c) = (f^{-1}(C))^c$

Pf: iPad deleted what I wrote; see textbook
Rmk: In general,
\( \{ \text{a} \} f(NCA) \neq \bigcap \{ f(A) : A \in cA \} \)
\( \{ \text{b} \} f(A^c) \neq (f(A))^c, \text{ for } A \subseteq X \)

Consider
\[ X = Y = \{ 1, 2, 3 \} \]
\[ f(x) = 1 \quad \forall x \in X \]
\[ cA = \{ \{ 13 \}, \{ 2 \} \} \rightarrow \text{this shows (a) above} \]
\[ A = \{ 3 \} \rightarrow \text{this shows (b) above} \]
**Def:** A binary operation on a set $X$ is a function from $X \times X \rightarrow X$.

**Def:** A **field** $F$ is a set with

A1. A binary operation called addition

A2. $(x+y) + z = x + (y+z)$ for all $x, y, z \in F$

A3. $x + y = y + x$ for all $x, y \in F$

A4. Existence of an additive identity $0$ such that $x + 0 = x$ for all $x \in F$

A5. Existence of additive inverses for all elements in $F$: For each $x \in F$, there exists an element $y \in F$ such that $x + y = 0$.

We will denote $y$ as $-x$.

**m1.** A binary operation called multiplication.

m2. $(xy)z = x(yz)$ for all $x, y, z \in F$

m3. $xy = yx$

m4. Existence of a multiplicative identity $1$ such that $x \cdot 1 = x$ for all $x \in F$

m5. For every $x \in F \setminus \{0\}$, there exists an element $y \in F$ such that $x \cdot y = 1$.

We will denote $y$ as $x^{-1}$ or $\frac{1}{x}$.

D. Distributivity: $x(y+z) = xy + xz$ for all $x, y, z \in F$.
Def: An ordered field is a field satisfying

(i) There is a subset \( P \) of \( F \) called the positive numbers satisfying
(ii) \( x \in F \) exactly one of the following holds:

\[ x \in P, \ x = 0, \ -x \in P \]

Given \( x, y \in F \)

- \( x \) is negative if \( -x \in P \)
- \( x > y \) if \( x - y \in P \)
- \( x \geq y \) if either \( x > y \) or \( y = x \)

Ex: \( \mathbb{R}, \mathbb{Q} \)
Def: Given an ordered field \( F \) and a nonempty set \( X \subseteq F \):

- \( a \in F \) is an upper bound of \( X \) if \( x \leq a \) for all \( x \in X \).
- \( X \) is bounded above if an upper bound exists.

- \( a \in F \) is the supremum of \( X \) if:
  (i) \( a \) is an upper bound for \( X \).
  (ii) for any upper bound \( a' \) of \( X \), \( a \leq a' \).

Def: The real numbers \( \mathbb{R} \) is the ordered field s.t.:

- \( \forall X \subseteq \mathbb{R}, X \neq \emptyset \), \( X \) bounded above, the supremum of \( X \) exists.

Thm: \( \mathbb{R} \) exists.
喉：“IR has no holes”

Def: The natural numbers $\mathbb{N}$ is the smallest subset of $\mathbb{R}$ having the properties that

(i) $1 \in \mathbb{N}$ and (ii) $n \in \mathbb{N} \implies n+1 \in \mathbb{N}$

D “smallest” is in the sense of set inclusion.

Thm:

(a) If $\mathcal{A} \subseteq 2^\mathbb{R}$ is the collection of $\mathcal{A} \subseteq \mathbb{R}$ s.t. (i) and (ii) hold, then for $\mathcal{A}$ satisfied (i) and (ii).

(b) $\mathbb{N} = \bigcap \mathcal{A}$. In particular, $\mathbb{N}$ exists.
(a) Since $A$ satisfies (i) $\forall x \in A, \ x \in \mathcal{A}$. If $m \in \mathcal{A}$, then $m \in A \land A \in A$, so $n + 1 \in A \land A \in A$. Hence $n + 1 \in \mathcal{A}$.

(b):