Lecture 2

Recall:
set X, power set 2^X

ordered pair (a,b)
Cartesian product
$$X \times Y = \{(a,b) : a \in X,\ b \in Y\}$$

ordered n-tuple $(a_1, a_2, \dots, a_n)$
Cartesian product
$$X_1 \times \dots \times X_n = \prod_{i=1}^{n} X_i$$
$$= \{(a_1, \dots, a_n) : a_i \in X_i,\ \forall i = 1, \dots, n\}$$

function $f: X \to Y$

$\boxed{f \subseteq X \times Y}$

- one to one / injective
- onto / surjective
- bijective

ordered field

- upper bound
- bounded above
- supremum / least upper bound

Def: The real numbers $\mathbb{R}$ is the ordered field s.t. $X \subseteq \mathbb{R}$, $X \neq \emptyset$, X bounded above, the supremum of X exists.

If M is the supremum of X, let $\sup(X) = M$.

Extend defn of sup~~remum~~

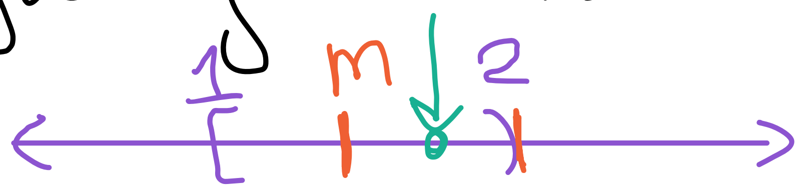
- If $X = \emptyset$, $\sup(X) := -\infty$
- If $X \neq \emptyset$ and is <u>unbounded above</u>, then $\sup(X) := +\infty$

In this way, $\forall\, X \subseteq \mathbb{R}$, $\sup(X)$ has meaning.

<u>Rmk</u>: The supremum of X D.N.E. $\iff \sup(X) = \pm\infty$.

Ex: $\sup([1,2)) = 2$

To justify this answer:



By defn of $[1,2)$,
$\forall x \in [1,2)$, $x < 2$, so 2 is an upper bound.

Assume, for the sake of contradiction that m is an upper bound of $[1,2)$ and $m < 2$. Since m is an upper bound of the set $m \geq 1$.
Let $x = \dfrac{m+2}{2}$. Then...

$$1 \leq m < x < 2.$$

Then $x \in [1,2)$ satisfies $x > m$, so m is not an upper bound. $\square$

Fact: $a, b \in \mathbb{R}$, $a < b + \varepsilon \ \forall \varepsilon > 0$, sufficiently small, then $a \le b$

To prove directly...

Let M be an upperbound of $[1,2)$. We must show $2 \le M$.

Fix $x \in [1,2)$. Then $x \le M$.

Thus $2 - \varepsilon \le M$ for all $\varepsilon > 0$ sufficiently small, so $2 \le M + \varepsilon \Rightarrow 2 \le M$.

Thm: $\mathbb{R}$ exists.

HW: $\mathbb{R}$ is unique, up to isomophism

Def: The natural numbers $\mathbb{N}$ is the smallest subset of $\mathbb{R}$ having the properties that
(i) $1 \in \mathbb{N}$
(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$

Thm:
(a) If $\mathcal{A} \subseteq 2^{\mathbb{R}}$ is the collection of $A \subseteq \mathbb{R}$ s.t. (i) and (ii) hold, then $\bigcap \mathcal{A}$ satisfies (i) and (ii).
(b) $\mathbb{N} = \bigcap \mathcal{A}$

<u>Rmk</u>: $\mathbb{N} = \{1,2,3,4,\dots\}$

- By definition, $1 \in A \ \forall A \in \mathcal{A}$.
- For any $n \in \{1,2,3,\dots\}$, if $n \in A$, then $n+1 \in A$ by (ii)

- By induction,

  $\{1,2,3,4,\dots\} \subseteq A \quad \forall A \in \mathcal{A}$

- Is it possible that $\{1,2,3,4,\dots\} \subsetneq \mathbb{N}$?
  No, since $\{1,2,3,4,\dots\} \in \mathcal{A}$.

More numbers!

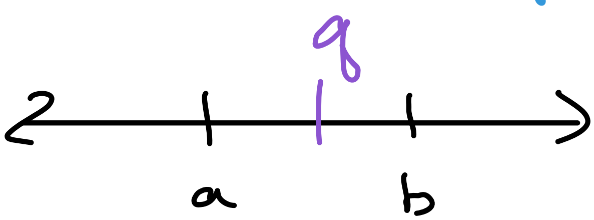
$-S = \{-s : s \in S\}$

$\mathbb{Z} = \{0\} \cup \mathbb{N} \cup -\mathbb{N}$

$\mathbb{Q} = \{p/q : p, q \in \mathbb{Z},\ q \neq 0\}$

$\mathbb{R}^d = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{d \text{ times}}$

<u>Thm</u> (Density of $\mathbb{Q}$ in $\mathbb{R}$):
$\forall a, b \in \mathbb{R}$ with $a < b$, $\exists q \in \mathbb{Q}$ s.t. $a < q < b$.



<u>Thm</u> (Archimedean Property)
$\forall a, b \in \mathbb{R}$, $a, b > 0$, $\exists n \in \mathbb{N}$ s.t. $na > b$.

bathtub

spoon

Chapter 3: ~~Set Equivalence~~ Cardinality

<u>Def</u>: Two nonempty sets X and Y have the same <u>cardinality</u> if there exists a bijection between them. We will write $|X| = |Y|$.

<u>Def</u>: For any $n \in \mathbb{N}$, write $|\{1, 2, 3, \ldots, n\}| = n$. $|\emptyset| = 0$.

Terminology: Given a set X (possibly empty)

- finite: $|X| \in \mathbb{N} \cup \{0\}$

infinite: if not finite

• countable: $|X| = |\mathbb{N}|$ or X is finite
uncountable: not countable

<u>Thm</u>: A nonempty set X is countable iff $\exists\, f: \mathbb{N} \to X$ that is surjective.

<u>Prop</u>: $\forall d \in \mathbb{N}$, $\mathbb{N}^d = \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{d \text{ times}}$

<u>Prop</u>: $\mathbb{Q}$ is countable

<u>Def</u>: Given $a, b \in \overbrace{\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}}^{\overline{\mathbb{R}}}$ an <u>interval</u> (between a and b)

is a set of the form:

- (a, b)
- $(a, b]$
- $[a, b)$
- $[a, b]$

$\{x : [-\infty, +\infty] = \overline{\mathbb{R}}$

<u>Prop</u>: For $a < b$, any interval between a and b is uncountable.

<u>Def</u>: A (real-valued) <u>sequence</u> is a function from $\mathbb{N}$ into $\mathbb{R}$.

<u>Rmk</u>: To emphacize that a sequence is a special type of real-valued function, instead of writing

$$f(n), \quad n \in \mathbb{N}$$

we will write

$$s_n, \quad n \in \mathbb{N}.$$

Often, we will abbreviate a sequence by listing its values

$$(s_1, s_2, s_3, s_4, \ldots), \quad n \in \mathbb{N}$$

$$(s_n)_{n \in \mathbb{N}} = (s_n)_{n=1}^{\infty} \cancel{= \{s_n\}_{n=1}^{\infty}}$$

Ex: $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

$\frac{1}{n}$

Ex: $(-1, 1, -1, -1, \ldots) = (-1)$

$\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1\}$

<u>Thm</u>: Let $A_1, A_2, \ldots$ be a countable family of countable sets. Then $\bigcup_{n=1}^{\infty} A_n$ is countable.

<u>Pf</u>: First, if $A_n = \emptyset \ \forall n \in \mathbb{N}$, the result is immediate, so we may assume $A_n \neq \emptyset$ for some $n \in \mathbb{N}$.

Next, if $A_m = \emptyset$ for $m \neq n$, then we may redefine $A_m := A_n$ without changing $\bigcup_{n=1}^{\infty} A_n$. Thus, we may assume that $A_n \neq \emptyset \ \forall n \in \mathbb{N}$.

Since A_n is countable for all $n \in \mathbb{N}$, we may list the elements as

$$a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots$$

← finite or infinite

If $|A_n| = k$, $k \in \mathbb{N}$, we define $a_\ell^{(n)} = a_k^{(n)} \ \forall \ell \in \mathbb{N}, \ell > k$.

Define $f(\ell, n) = a_\ell^{(n)}$. Then $f: \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ is surjective.

Thus, $\bigcup_{n=1}^{\infty} A_n$ is countable. $\square$