Lecture 3

Recall:

<u>Thm</u> (Well-Ordering): If X is a nonempty subset of $\mathbb{N}$, $\min(X)$ exists.

<u>Thm</u> (Archimedean Property)
$\forall a, b \in \mathbb{R},\ a, b > 0,\ \exists\, n \in \mathbb{N}$ s.t. $na > b$.

bathtub

spoon

<u>Thm</u> (Density of $\mathbb{Q}$ in $\mathbb{R}$):
$\forall a, b \in \mathbb{R}$ with $a < b$, $\exists\, q \in \mathbb{Q}$ s.t. $a < q < b$.

Def: Two nonempty sets X and Y have the same cardinality if there exists a bijection between them. We will write $|X| = |Y|$.

Def: For any $n \in \mathbb{N}$, write $|\{1, 2, 3, \ldots, n\}| = n$. $|\emptyset| = 0$.

Terminology: Given a set X possibly empty

- finite: $|X| \in \mathbb{N} \cup \{0\}$
  infinite: if not finite

- countable: $|X| = |\mathbb{N}|$ or X is finite
  uncountable: not countable

<u>Thm</u>: Any subset of $\mathbb{N}$ is countable.

Pf: Suppose $X \subseteq \mathbb{N}$. If X is finite, the result is immediate.

Suppose X is infinite. Define $f: \mathbb{N} \to X$ as follows:

$f(1) = \min(X)$

$f(2) = \min(X \setminus \{f(1)\})$

$\vdots$

$f(n) = \min(X \setminus \{f(1), f(2), \dots, f(n-1)\})$

By definition f is strictly increasing, $n < m \Leftrightarrow f(n) < f(m)$.

Thus f is injective.

Furthermore, for any $x \in X \subseteq \mathbb{N}$, there can be $\underbrace{\text{at most finitely}}_{\text{at most } x}$ many $y \in X$ s.t. $y < x$.

Thus $f(n) = x$ for some $n \leq x$.
Thus f is surjective.
Therefore $|X| = |\mathbb{N}|$. $\square$

Cor: Any subset of a countable set is countable.

Pf: Suppose Y is countable and $X \subseteq Y$. If Y finite, so is X, hence the result holds. Suppose $|Y| = |\mathbb{N}|$. Then $\exists\, f: Y \to \mathbb{N}$ bijective. Since $f(X) \subseteq \mathbb{N}$, $f(X)$ is countable. Since $f: X \to f(X)$ is a bijection, $|X| = |f(X)|$, so is countable.
$f(X) = \{f(x) : x \in X\}$

Thm: A nonempty set X is countable iff $\exists\, f: \mathbb{N} \to X$ that is surjective.

Pf:
Suppose X is countable. If $|X| = |\mathbb{N}|$, $\exists\, f: \mathbb{N} \to X$ bijective, so the result holds.

Now, suppose $|X| = k$ for $k \in \mathbb{N}$. $X = \{x_1, x_2, \ldots, x_k\}$. Let $f: \mathbb{N} \to X$ be $f(\ell) = x_{\min(\ell, k)}$. This is surjective, which gives the result.

Now, assume $\exists\, f: \mathbb{N} \to X$ surjective. If X is finite, the result is immediate. Assume X is infinite.

Define $g: \mathcal{X} \to \mathbb{N}$ by $g(x) = \min(f^{-1}(\{x\}))$

Note that g is injective, since

$$
\begin{aligned}
g(x) = g(y) &\iff \min(f^{-1}\{x\}) = \min(f^{-1}\{y\}) \\
&\implies f^{-1}(\{x\}) \cap f^{-1}(\{y\}) \neq \emptyset \\
&\iff \exists\, z \text{ s.t. } f(z) = x \text{ and } f(z) = y \\
&\implies x = y
\end{aligned}
$$

Thus, g is surjective onto $g(\mathcal{X})$. Hence $|\mathcal{X}| = |g(\mathcal{X})|$. Since $g(\mathcal{X}) \subseteq \mathbb{N}$, $g(\mathcal{X})$ is countable. Thus $\mathcal{X}$ is countable.

Will return   Assume $X \neq \emptyset$

Q: If $\exists\ g: X \to \mathbb{N}$ that is injective, is X countable?

$\Updownarrow$

Q: If $\exists\ g: X \to \mathbb{N}$ that is injective, does there exist $f: \mathbb{N} \to X$ that is surjective.

A: Yes!

Pf: Since $g: X \to g(X)$ is a bijection, $\forall\ y \in g(X)$ we may define

$$f(y) = g^{-1}(y)$$

Then $f: g(X) \to X$ is a bijection.

Fix $x_0 \in X$. If $y \in \mathbb{N} \setminus g(X)$, let $f(y) = x_0$.

Then $f: \mathbb{N} \to X$ is surjective.

Prop: $\forall d \in \mathbb{N}$, $\mathbb{N}^d = \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{d \text{ times}}$ is countable.

Prop: $\mathbb{Q}$ is countable

Def: Given $a, b \in \overbrace{\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}}^{\overline{\mathbb{R}}}$, an interval (between a and b) is a set of the form:

- (a, b)
- $(a, b]$
- $[a, b)$
- $[a, b]$

$$\{x : [-\infty, +\infty] = \overline{\mathbb{R}}$$

Prop: For $a<b$, any interval between a and b is uncountable.

Def: A (real-valued) **sequence** is a function from $\mathbb{N}$ into $\mathbb{R}$, written s_n for $n \in \mathbb{N}$.

Thm: Let $A_1, A_2, \ldots$ be a countable family of countable sets. Then $\bigcup_{\cancel{n\in\mathbb{N}}} A_n$ is countable.

If finitely many $A_1, A_2, \ldots, A_k$, define $A_\ell = A_k$ for all $\ell \in \mathbb{N}$ with $\ell \geq k$. WLOG infinitely many A_n

Pf: First, if $A_n = \emptyset \ \forall n \in \mathbb{N}$, the result is immediate, so we may assume $A_n \neq \emptyset$ for some $n \in \mathbb{N}$.

Next, if $A_m = \emptyset$ for $m \neq n$, then we may redefine $A_m := A_n$ without changing $\bigcup_{n=1}^{\infty} A_n$. Thus, we may assume that $A_n \neq \emptyset \ \forall n \in \mathbb{N}$.

Since A_n is countable for all $n \in \mathbb{N}$, we may list the elements as

$$a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots$$ ← finite or infinite

If $|A_n| = k$, $k \in \mathbb{N}$, we define $a_\ell^{(n)} = a_k^{(n)}$ $\forall \ell \in \mathbb{N}, \ell > k$.

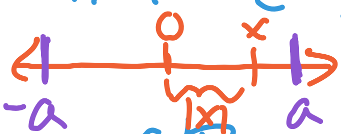
Define $f(\ell, n) := a_\ell^{(n)}$. Then $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{\cancel{n=1}} A_n$ is surjective.

Since $\mathbb{N}\times\mathbb{N}$ is countable, $\exists$ a bijection $g:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$. Thus $g\circ f:\mathbb{N}\to\bigcup_{n=1}^{\infty}A_n$ is surjective.

Thus, $\bigcup_{n=1}^{\infty}A_n$ is countable. $\square$

Recall: Properties of Absolute Value

Def: For $x\in\mathbb{R}$, $|x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & x<0 \end{cases}$



Thm: For all $x, y, a\in\mathbb{R}$, $a>0$

(i) $|x|<a \iff -a<x<a$

(ii) $x\leq |x|$, $-x\leq |x|$

(iii) $|xy|=|x||y|$

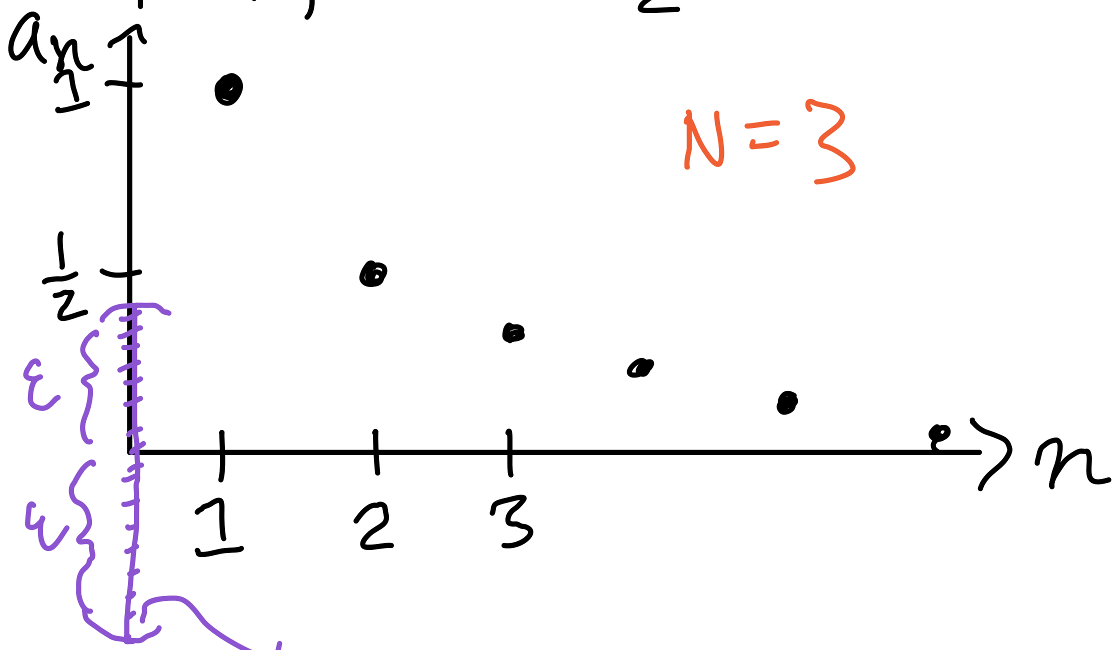
(iv) $|x+y|\leq |x|+|y|$ triangle ineq.

Ch 4: Sequences of Real Numbers

Section 10: Sequences

Ex:

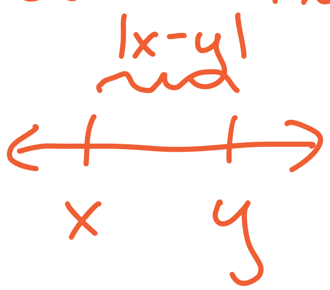
- $a_n = \frac{1}{n}$, $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$
- $b_1 = 1$, $b_n = \frac{b_{n-1}}{2}$



collection of points within distance ε from zero

Def: A sequence a_n <u>converges</u> to $L \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$ ensures $|a_n - L| < \varepsilon$.
We call L the <u>limit</u> of a_n and write $\lim_{n \to \infty} a_n = L$.

Recall: $|x-y|$ = distance between x and y



Def: A sequence that does not converge to any $L \in \mathbb{R}$ <u>diverges</u>.

"~~has no limit~~"

Ex: $a_n = \frac{n}{n+1}$, $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right.$

We will prove $\lim_{n\to\infty} a_n = 1$. Fix arbitrary $\varepsilon > 0$.

Scratchwork:

$$|a_n - 1| < \varepsilon \iff \left|\frac{n}{n+1} - 1\right| < \varepsilon$$
$$\iff \left|\frac{n-(n+1)}{n+1}\right| < \varepsilon$$
$$\iff \left|\frac{-1}{n+1}\right| < \varepsilon$$
$$\iff \frac{1}{n+1} < \varepsilon$$
$$\iff \frac{1}{\varepsilon} < n+1$$

Choose $N > \frac{1}{\varepsilon}$. Then $n \geq N$, $n+1 \geq N > \frac{1}{\varepsilon}$, so by scratchwork above, $|a_n - 1| < \varepsilon$.

Section 11: Subsequences