Recall: Properties of Absolute Value

**Def:** For \( x \in \mathbb{R} \), \(|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \)

**Thm:** For all \( x, y, a \in \mathbb{R} \), \( a > 0 \)

1. \(|x| < a \iff -a < x < a\)
2. \( x \leq |x| \)
3. \(|xy| = |x||y|\)
4. \(|x+y| \leq |x| + |y|\)
5. \(|x-y| \leq |x| + |y|\)

**Triangle Ineq**

**Lemma:** For all \( x, y, a \in \mathbb{R} \), \( a > 0 \),

\(|x-y| < a \iff y-a < x < y+a\)
Facts (HW3): Given $x, y \in \mathbb{R}$,

- $x \leq y + 3 \quad \forall x \in \mathbb{R}, 3 \geq 0 \Rightarrow x \leq y$
- $|x - 1| - |y| \leq |x - y|$

Reverse triangle ineq

Ch 4: Sequences of Real Numbers

Section 10: Sequences

Def: A sequence $a_n$ converges to $L \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$ ensures $|a_n - L| < \varepsilon$. We call $L$ the limit of $a_n$ and write $\lim_{n \to \infty} a_n = L$. 
Def: A sequence that does not converge to any $L \in \mathbb{R}$ diverges.

Ex: $b_n = (-1)^n, \quad (-1, 1, -1, 1, \ldots)$

Pf: Assume, for the sake of contradiction, that $b_n$ converges to some $L \in \mathbb{R}$. 

Ex: $S_n = \frac{(-1)^n}{n}$
Then, for $\varepsilon = \frac{1}{2}$, $\exists \ N \in \mathbb{N}$

s.t. $n \geq N$ ensures $|b_n - L| < \frac{1}{2}$

$\iff b_n - \frac{1}{2} < L < b_n + \frac{1}{2}$.

Since $\exists \ n$ even with $n \geq N$,

$1 - \frac{1}{2} = \frac{1}{2} < L$. Since $\exists \ n$ odd

with $n \geq N$, $L < -\frac{1}{2} = (-1)^{+\frac{1}{2}}$.

This is a contradiction.

**Thm:** The limit of a sequence

is unique.

**Pf:** Suppose an converges to both $L, L' \in \mathbb{R}$. Fix $\varepsilon > 0$ arbitrary.
Then $\exists N, N'$ s.t. "3/2 style argument"

- $n \geq N \implies |a_n - L| < 3/2$
- $n \geq N' \implies |a_n - L'| < 3/2$

Let $\bar{N} = \max \{N, N'\}$. Then $\forall n \geq \bar{N},$

"add and subtract"

$$|L-L'| = |L-a_n + a_n - L'|$$

$$= |(L-a_n) - (L'-a_n)|$$

$$\leq |L-a_n| + |L'-a_n|$$

$$< \frac{3}{2} + \frac{3}{2}$$

$$= 3$$

Since $\varepsilon > 0$ was arbitrary, by Fact, $|L-L'| = 0 \implies L = L'$. \qed
An equivalent defn of convergence...

**Def:** A sequence \( an \) converges to \( L \in \mathbb{R} \) if, \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \) s.t. \( n \geq N \) ensures \( |a_n - L| < \varepsilon \).

We call \( L \) the limit of \( an \) and write \( \lim_{n \to \infty} a_n = L \).

**Alt Def:** A sequence \( an \) converges to \( L \in \mathbb{R} \) if, \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \) holds for at most finitely many \( n \in \mathbb{N} \).
Remark:

- Sometimes we will consider $S_n$ that are only defined for $n$ sufficiently large.
  \[ \exists N \text{ s.t. } n \geq N \]
  Its limit is still well-defined.

  \[ \lim_{n \to \infty} S_n = \frac{1}{n-5} \]

- We may modify finitely many elements in a sequence, and whether it converges or diverges, the limiting behavior does not change.
Def: Given a sequence $s_n$, it is...

- increasing, in case $n \leq m \Rightarrow s_n \leq s_m$
- strictly increasing, in case $n < m \Rightarrow s_n < s_m$
- decreasing, in case $n \leq m \Rightarrow s_n \geq s_m$
- strictly decreasing, in case $n < m \Rightarrow s_n < s_m$

Ex: $\frac{1}{n}, \ 1 - n^{-1}$

Def: Given a sequence $s_n$, for any strictly increasing sequence $n_k$ of natural numbers, a sequence of the form $s_{n_k}$ is a subsequence of $s_n$. 
Remark: We could write \( s_n \) as \( s(n) \), \( n_k \) as \( n(k) \), and \( s_{n_k} \) as \( s(n(k)) \).

Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order.

Ex: \( s_n = (-1, 2, -3, 4, -5, ...) \)

\( s_{n_k} = (-1, -3, -5, ...) \)

\( n_k = (1, 3, 5, 7, ...) \)
Lemma: If \( n_k \) is a strictly increasing sequence of natural numbers, then \( n_k \geq k \) \( \forall k \in \mathbb{N} \).

**Proof:** We proceed by induction.

**Base case:** \( k = 1 \), since \( n_1 \in \mathbb{N}, n_1 \geq 1 \).

**Inductive step:** Suppose \( n_{k-1} \geq k-1 \). Then \( n_k \geq n_{k-1} \). Since \( n_k, n_{k-1} \in \mathbb{N} \), \( n_k \geq n_{k-1} + 1 \geq k \).

**Thm:** If a sequence \( s_n \) converges to a limit \( L \in \mathbb{R} \), then every subsequence also converges to \( L \).
Pf: Fix an arbitrary subsequence $S_{n_k}$ of $S_n$. Fix arbitrary $\varepsilon > 0$. Since $\lim_{n \to \infty} S_n = L$, there exists $N$ s.t. $\forall n \geq N$ ensures $|S_n - L| < \varepsilon$. If $k \geq N$, then by the previous lemma, $n_k \geq N$, so $|S_{n_k} - L| < \varepsilon$. This shows $\lim_{k \to \infty} S_{n_k} = L$.

Ex: $(-1)^n$ diverges, since subsequences of even and odd elements have different limits.

Ex: the constant sequence $a_n = (2, 2, 2, \ldots)$ converges to $2$. 

Thm (Limit of Sum is Sum of Limit):
If \( a_n \) and \( b_n \) are convergent sequences, so is \( a_n + b_n \) and
\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n.
\]

**Pf:** Fix \( \varepsilon > 0 \). Since \( a_n \) and \( b_n \) converge, \( \exists \ N_a, N_b \in \mathbb{N} \ s.t.
\[
\begin{align*}
    n \geq N_a & \implies |a_n - L| < \frac{\varepsilon}{2} \\
    n \geq N_b & \implies |b_n - M| < \frac{\varepsilon}{2}.
\end{align*}
\]

Let \( N = \max \{ N_a, N_b \} \). Then \( n \geq N \implies 
\[
|a_n + b_n - (L + M)| \\
= |(a_n - L) + (b_n - M)| \\
\leq |a_n - L| + |b_n - M| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon.
\]