Lecture 5

- Office hours: today, 3:30-4:30pm (not tomorrow, email questions)
- No class Wed; next class Mon Apr 22

Thm: The limit of a sequence is unique.

An equivalent defn of convergence...

Alt Def: A sequence a_n converges to $L \in \mathbb{R}$ if, $\forall\, \varepsilon > 0$, $|a_n - L| \geq \varepsilon$ holds for at most finitely many $n \in \mathbb{N}$.

Remark:

- Sometimes we will consider s_n that are only defined for $\underbrace{n \text{ sufficiently large}}_{\exists N \text{ s.t. } n \geq N \ldots}$

  Its limit is still well-defined.

- We may modify finitely many elements in a sequence, and the $\underbrace{\text{limiting behavior}}_{\text{whether it converges or diverges}}$ does not change.

11 Subsequences

Def: Given a sequence s_n, it is...

- increasing, in case $n \le m \Rightarrow s_n \le s_m$
- strictly increasing, in case $n < m \Rightarrow s_n < s_m$
- decreasing, in case $n \le m \Rightarrow s_n \ge s_m$
- strictly decreasing, in case $n < m \Rightarrow s_n > s_m$
- monotone, in case either increasing or decreasing.

Def: Given a sequence s_n, for any strictly increasing sequence n_k of natural numbers, a sequence of the form s_{n_k} is a subsequence of s_n.

Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order.

$$\text{Ex: } s_n = (\overset{n=1}{0}, \overset{n=2}{-1}, 0, \dots, \cos\left(\tfrac{n\pi}{2}\right), \dots)$$

$$s_{n_k} = (\overset{k=1}{-1}, \overset{k=2}{1}, -1, 1, \dots, (-1)^k, \dots)$$

$$n_k = (\overset{k=1}{2}, \overset{k=2}{4}, \dots, 2k, \dots)$$

<u>Lemma</u>: If n_k is a strictly increasing sequence of naturals numbers, then $n_k \geq k$ $\forall k \in \mathbb{N}$.

Thm: If a sequence s_n converges to a limit $L \in \mathbb{R}$, then every subsequence also converges to L.

12 The Algebra of Limits

Thm (Limit of Sum is Sum of Limit): If a_n and b_n are convergent sequences, so is $a_n + b_n$ and

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Thm: If $c \in \mathbb{R}$ and a_n is a convergent sequence, so is ca_n and

$$\lim_{n\to\infty} ca_n = c \underbrace{\lim_{n\to\infty} a_n}_{L}.$$

$$|ca_n - cL| < \varepsilon \iff |c||a_n - L| < \varepsilon$$

$$\iff |a_n - L| < \frac{\varepsilon}{|c|} \quad c \neq 0$$

Pf: If $c=0$, ca_n is the constant sequence $(0,0,0,\ldots)$, and the result is immediate.

Suppose $c \neq 0$. Fix $\varepsilon > 0$. Since a_n converges, $\exists$ N s.t. $n \geq N$ ensures
$|a_n - L| < \frac{\varepsilon}{|c|} \iff |ca_n - cL| < \varepsilon$.
Thus $\lim_{n\to\infty} ca_n = cL$.

Lemma: If a_n and b_n converge to 0, then $a_n b_n$ converges to 0.

$$|a_n b_n - 0| < \varepsilon \iff |a_n b_n| < \varepsilon$$
$$\iff |a_n||b_n| < \varepsilon$$

Pf: Fix $\varepsilon > 0$. Then $\exists\ N_a, N_b$ s.t.

$n \geq N_a \Rightarrow |a_n| < 1$;

$n \geq N_b \Rightarrow |b_n| < \varepsilon$.

Let $N = \max\{N_a, N_b\}$. Then $n \geq N$ ensures

$$|a_n b_n| = |a_n||b_n| \leq 1 \cdot |b_n| < \varepsilon$$

Thm (Limit of Product is Product of Lim)

If a_n and b_n converge, then so does $a_n b_n$ and

$$\lim_{n\to\infty} a_n b_n = \underbrace{\left(\lim_{n\to\infty} a_n\right)}_{L}\underbrace{\left(\lim_{n\to\infty} b_n\right)}_{M}.$$

Pf: Note that

$$a_n b_n = \underbrace{(a_n - L)}_{0}\underbrace{(b_n - M)}_{0} + \underbrace{a_n M}_{LM} + \underbrace{b_n L}_{LM} - \underbrace{LM}_{-LM}$$

converging to...

...where we interpret the sequence

$a_n - L = (a_1 - L, a_2 - L, a_3 - L, \ldots)$

Note that $\lim_{n\to\infty} a_n - L = 0$.

Since the limit of sum is sum of limits,

$\lim_{n\to\infty} a_n b_n = 0 + LM + LM - LM = LM.$

<u>Lemma</u>: Suppose $\lim_{n\to\infty} a_n = L \in \mathbb{R}\setminus\{0\}$.
Then

- $a_n \neq 0$, for all but finitely many n
- $\frac{1}{a_n}$ converges
- $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$.

Scratch:

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| < \varepsilon \iff \left|\frac{L - a_n}{L a_n}\right| < \varepsilon$$

$$\iff \frac{|L - a_n|}{|L a_n|} < \varepsilon$$

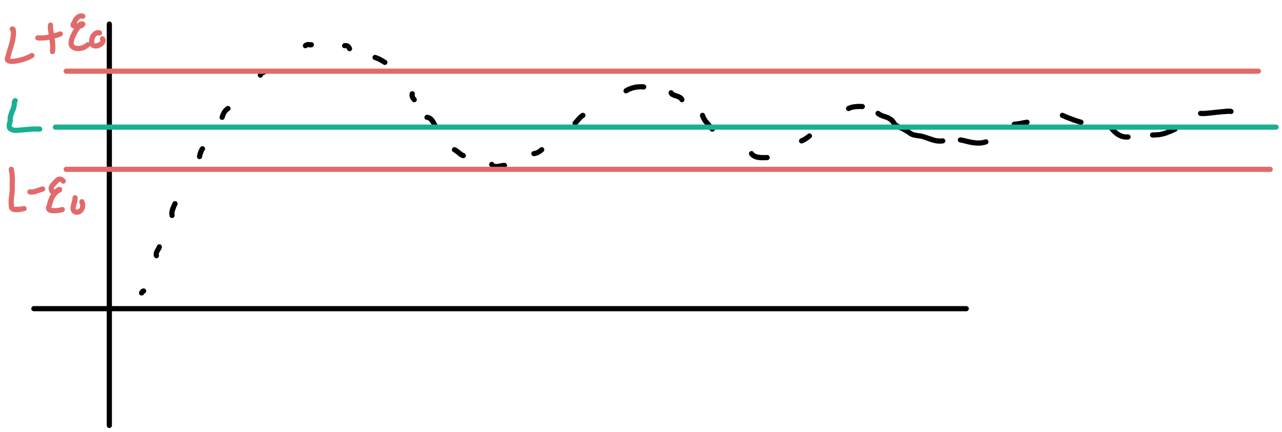
is it strict?

$$\iff |L - a_n| < \varepsilon |L||a_n|$$

$$\Longleftarrow \quad |L - a_n| < \varepsilon |L| \frac{|L|}{2}$$

reverse $\triangle$ ineq

$$|L| - |a_n| \leq |a_n - L| < \frac{|L|}{2} \implies \frac{|L|}{2} < |a_n|$$



Pf: Fix $\varepsilon > 0$. Since a_n converges to $L \neq 0$, $\varepsilon |L| \frac{|L|}{2} \neq 0$, so $\exists\, N_1$ s.t. $n \geq N_1$ ensures
$|a_n - L| < \frac{\varepsilon |L||L|}{2}$.

Furthermore, $\exists\, N_2$ s.t. $n \geq N_2$ ensures

reverse Δ

$$|L| - |a_n| \leq |a_n - L| < \frac{|L|}{2},$$

Thus, $\frac{|L|}{2} < |a_n|$. This shows $a_n \neq 0$, for all but finitely many n.

Furthermore, for $N = \max\{N_1, N_2\}$, $n \geq N$ ensures
$$|a_n - L| < \frac{\varepsilon |L||L|}{2} < \varepsilon |L||a_n|$$
Thus, $n \geq N$ ensures $\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|L||a_n|} < \varepsilon$.

Cor: If a_n and b_n are convergent sequences and $\lim_{n\to\infty} b_n \neq 0$, then $\frac{a_n}{b_n}$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Ex: $\lim_{n\to\infty} \frac{2n}{n+2} = \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}}$

- Since $\frac{1}{n}$ converges to 0, $\lim_{n\to\infty} \frac{2}{n} = 0$
- Since limit of sum is sum of limits, $\lim_{n\to\infty} 1 + \frac{2}{n} = 1 + 0 = 1$
- Since limit of reciprocal is reciprocal of limit, $\lim_{n\to\infty} \frac{1}{1+2/n} = \frac{1}{1} = 1$.
- $\lim_{n\to\infty} 2 \cdot \left(\frac{1}{1+2/n}\right) = 2 \cdot 1 = 2$.

13 Bounded Sequences

Def: A sequence a_n is

- **bounded above** if $\exists\, M \in \mathbb{R}$ s.t. $a_n \le M \quad \forall n \in \mathbb{N}$
- **bounded below** if $\exists\, m \in \mathbb{R}$ s.t. $a_n \ge m \quad \forall n \in \mathbb{N}$
- **bounded** if it is bounded above and below

$\Updownarrow$

if $\exists\, M \ge 0$ s.t. $\overbrace{|a_n| \le M}^{-M \le a_n \le M} \quad \forall n \in \mathbb{N}$.

Thm: Convergent sequences are bounded.

Pf: Suppose a_n converges to L.
Then $\exists N$ s.t. $n \geq N$ ensures
$|a_n - L| < 1 \implies |a_n| - |L| < 1$

reverse Δ

$\iff |a_n| < |L| + 1$

Let $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |L|+1\}$.
Then $|a_n| \leq M \quad \forall n \in \mathbb{N}$.