Lecture 6

- Solution posted to HW2, Q10
- Makeup lecture this Friday 3:30-4:45pm

12 The Algebra of Limits

Thm (Limit of Sum is Sum of Limit):
If $a_n$ and $b_n$ are convergent sequences, so is $a_n + b_n$ and
\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n.
\]

Thm: If $c \in \mathbb{R}$ and $a_n$ is a convergent sequence, so is $ca_n$ and
\[
\lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n.
\]
Thm (Limit of Product is Product of Limits)

If \( a_n \) and \( b_n \) converge, then so does \( a_n b_n \) and
\[
\lim_{n \to \infty} a_n b_n = (\lim_{n \to \infty} a_n)(\lim_{n \to \infty} b_n).
\]

Cor: If \( a_n \) and \( b_n \) are convergent sequences and \( \lim_{n \to \infty} b_n \neq 0 \), then \( \frac{a_n}{b_n} \) converges and
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}.
\]
Def: A sequence an is
• bounded above if \( \exists M \in \mathbb{R} \text{s.t.} \ an \leq m \ \forall n \in \mathbb{N} \)
• bounded below if \( \exists M \in \mathbb{R} \text{s.t.} \ an \geq m \ \forall n \in \mathbb{N} \)
• bounded if it is bounded above and below
  \[ \exists m \leq an \leq M \]
if \( \exists M \geq 0 \text{s.t.} \ |an| \leq M \ \forall n \in \mathbb{N} \).

Thm: Convergent sequences are bounded.
Ex: \( n^2 \sin(\pi n/2) (-1)^n \)

**Thm:** If \( a_n \) is bounded and  
\[ \lim_{n \to \infty} b_n = 0, \]  
then \( \lim_{n \to \infty} a_n b_n = 0. \)

**Scratch:**  
\[ |a_n b_n - 0| < \epsilon \iff |a_n||b_n| < \epsilon \]
\[ \iff M|b_n| < \epsilon \]
\[ \iff |b_n| < \frac{\epsilon}{M} \]

**Pf:** Fix \( \epsilon > 0 \). Since \( a_n \) is bounded  
\( \exists M \geq 0 \) s.t. \( |a_n| \leq M \), \( \forall n \in \mathbb{N} \)  
\[ \text{wlog } M > 0 \]
Since \( \lim_{n \to \infty} b_n = 0 \), \( \exists N \) s.t. \( n \geq N \) ensures \( |b_n| < \frac{1}{m} \Leftrightarrow M |b_n| < \varepsilon \). Thus \( |a_n b_n - 0| = |a_n| |b_n| < \varepsilon \). Hence \( \lim_{n \to \infty} a_n b_n = 0 \).

### Further Limit Theorems

**Thm:** Suppose \( a_n, b_n \) are convergent sequences with \( a_n \leq b_n \) for all but finitely many \( n \leq N \). Then \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \).

*Ref.: HW4*
**Thm (Squeeze):** Suppose 
\[ a_n \leq b_n \leq c_n \]
for all but finitely many \( n \in \mathbb{N} \) and 
\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \in \mathbb{R} \].

Then 
\[ \lim_{n \to \infty} b_n = L \]
15 Divergent Sequences

From now on, consider $a_n: \mathbb{N} \to \mathbb{R}$. The same definition of convergence applies to such sequences. However, if $a_n$ converges, $a_n \in \mathbb{R}$ for all but finitely many $n \in \mathbb{N}$.

Def: $\forall n \in \mathbb{N}$

- $a_n$ diverges to $+\infty$ if, $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $n \geq N$ ensures $a_n > M$.

We write $\lim_{n \to \infty} a_n = +\infty$. 
\( \begin{align*}
&\text{an diverges to } -\infty \text{ if, } \forall M \in \mathbb{R}, \\
&\exists N \text{ s.t. } n \geq N \text{ ensured } a_n < M. \\
&\text{We write } \lim_{n \to \infty} a_n = -\infty.
\end{align*} \)

Q: divergence to \( \pm \infty \Rightarrow \text{diverges?} \)
Def: Given a sequence \( a_n \), the limit exists if \( a_n \) converges or \( a_n \) diverges to \( \pm \infty \).

\[
\lim_{n \to \infty} a_n \in \mathbb{R}.
\]

Thm (Squeeze): Suppose \( a_n \leq b_n \) for all but finitely many \( n \) and the limits exist. Then \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \).

Pf: On HW4, you will show the result if both sequences converge.
If $a_n$ diverges to $-\infty$ or $b_n$ diverges to $+\infty$, the result is clearly true.

Suppose $a_n$ diverges to $+\infty$. We will show $b_n$ diverges to $+\infty$.

Fix $M \in \mathbb{R}$. $\exists N \text{ s.t. } n \geq N$ ensured $M < a_n$. Choose $N'$ so that $a_n \leq b_n$ $\forall n \geq N'$.

Let $N'' = \max\{N, N'\}$. Then $n \geq N''$ ensured $M < a_n \leq b_n$.

Thus, $\lim_{n \to \infty} b_n = +\infty$.

It remains to show $\lim_{n \to \infty} b_n = -\infty$.

$\implies \lim_{n \to \infty} b_n = -\infty$. See HW5.
Recall:
increasing \( a_n \leq a_{n+1} \), \( \forall n \in \mathbb{N} \)
decreasing \( a_n \geq a_{n+1} \), \( \forall n \in \mathbb{N} \)
monotone if either increasing or decreasing

Remark:
increasing sequences are bold below as long as \( a_1 \neq -\infty \)
decreasing sequences are bold above as long as \( a_1 \neq +\infty \).
Thm: All bounded monotone sequences converge.

Proof: Suppose \( a_n \) is bounded and increasing. Since \( a_n \) is bounded, \( \{a_n : n \in \mathbb{N}\} \) is bounded above, its supremum exists. Let \( L = \sup \{a_n : n \in \mathbb{N}\} \).

Fix \( \varepsilon > 0 \). Since \( L \) is an upper bound, \( L \geq a_n \) for all \( n \in \mathbb{N} \).
Since \[ L - \varepsilon < L, \quad L - \varepsilon \text{ is not an upper bound, so } \exists N \text{ s.t. } a_N > L - \varepsilon. \quad \text{Since } a_n \text{ is increasing, } a_n \geq a_N > L - \varepsilon \quad \text{for all } n \geq N. \]

Thus \[ n \geq N, \quad L - \varepsilon < a_n \leq L < L + \varepsilon \]
\[ \Rightarrow |a_n - L| < \varepsilon. \]

Now, suppose \( a_n \) is bounded and decreasing. Then \( -a_n \) is bounded and increasing, so it converges to \( \pm \infty \).

Thus \[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)(-a_n) = -L. \]
**Ex**

**Claim:** If \(|a| < 1\), \(\lim_{n \to \infty} a^n = 0\).

**Proof of Claim:**

If \(a = 0\), the result is immediate.

Suppose \(0 < a < 1\).

1) \(a^n\) is decreasing (by induction)
2) \(a^n\) is bounded, since product of nonneg and decreasing.

Thus, \(\lim_{n \to \infty} a^n = 0\).
Since \( a^{n+1} \) is a subseq of \( a^n \)
\[
L = \lim_{n \to \infty} a^{n+1} = \lim_{n \to \infty} a a^n = a L.
\]

If \( L \neq 0 \), then \( 1 = \alpha \), which is a contradiction. Thus \( L = 0 \).

Suppose \( -1 < \alpha < 0 \).
Then \( \lim_{n \to \infty} (-\alpha)^n = 0 \). Thus
\[
\lim_{n \to \infty} a^n = \lim_{n \to \infty} (-1)^n (-\alpha)^n = 0. \quad \square
\]