Lecture 6

- Solution posted to HW2, Q10
- Makeup lecture this Friday 3:30-4:45pm

12 The Algebra of Limits

Thm (Limit of Sum is Sum of Limit): If a_n and b_n are convergent sequences, so is a_n+b_n and

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Thm: If $c \in \mathbb{R}$ and a_n is a convergent sequence, so is ca_n and $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$.

Thm (Limit of Product is Product of Lim)
If a_n and b_n converge, then so does $a_n b_n$ and

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

Cor: If a_n and b_n are convergent sequences and $\lim_{n\to\infty} b_n \neq 0$, then $\frac{a_n}{b_n}$ converges and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

13 Bounded Sequences

<u>Def</u>: A sequence a_n is

- <u>bounded above</u> if $\exists\, M \in \mathbb{R}$ s.t. $a_n \le M \quad \forall n \in \mathbb{N}$
- <u>bounded below</u> if $\exists\, m \in \mathbb{R}$ s.t. $a_n \ge m \quad \forall n \in \mathbb{N}$
- <u>bounded</u> if it is bounded above and below

$\Updownarrow$

if $\exists\, M \ge 0$ s.t. $\overbrace{|a_n| \le M}^{-M \le a_n \le M} \quad \forall n \in \mathbb{N}$.

<u>Thm</u>: Convergent sequences are bounded.

Ex: n^2

$\sin(\pi n/2)$

$(-1)^n$

Thm: If a_n is bounded and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.

Scratch:

$$|a_n b_n - 0| < \varepsilon \iff |a_n||b_n| < \varepsilon$$
$$\Leftarrow M|b_n| < \varepsilon$$
$$\iff |b_n| < \varepsilon/M$$

Pf: Fix $\varepsilon > 0$. Since a_n is bounded $\exists\ \underbrace{M \geq 0}_{\text{WLOG } M>0}$ s.t. $|a_n| \leq M$, $\forall n \in \mathbb{N}$

Since $\lim_{n\to\infty} b_n = 0$, $\exists N$ s.t. $n \geq N$ ensures $|b_n| < \varepsilon/M \Leftrightarrow M|b_n| < \varepsilon$.
Thus $|a_n b_n - 0| = |a_n||b_n| < \varepsilon$.
Hence $\lim_{n\to\infty} a_n b_n = 0$.

14 Further Limit Theorems

Thm: Suppose a_n, b_n are convergent sequences with $a_n \leq b_n$ for all but finitely many $n \in \mathbb{N}$.
Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

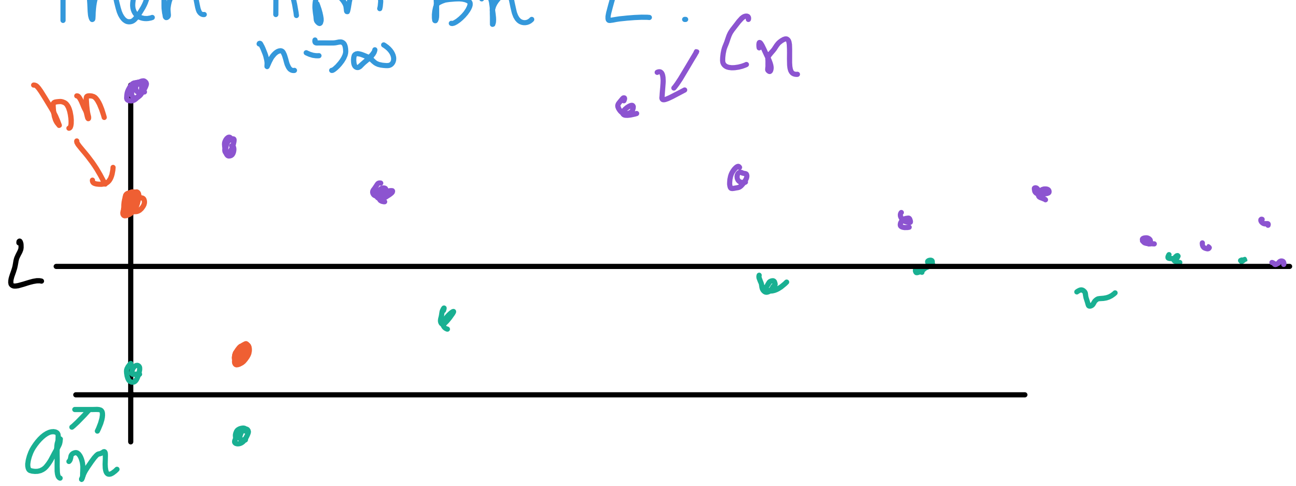
Pf: HW4

Thm (Squeeze): Suppose

$$a_n \le b_n \le c_n$$

for all but finitely many $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \in \mathbb{R}$.

Then $\lim_{n\to\infty} b_n = L$.



15 Divergent Sequences

From now on, consider $a_n : \mathbb{N} \to \overline{\mathbb{R}}$.

resp. divergence

The same definition of convergence to limit $L \in \overline{\mathbb{R}}$ applies to such sequences.

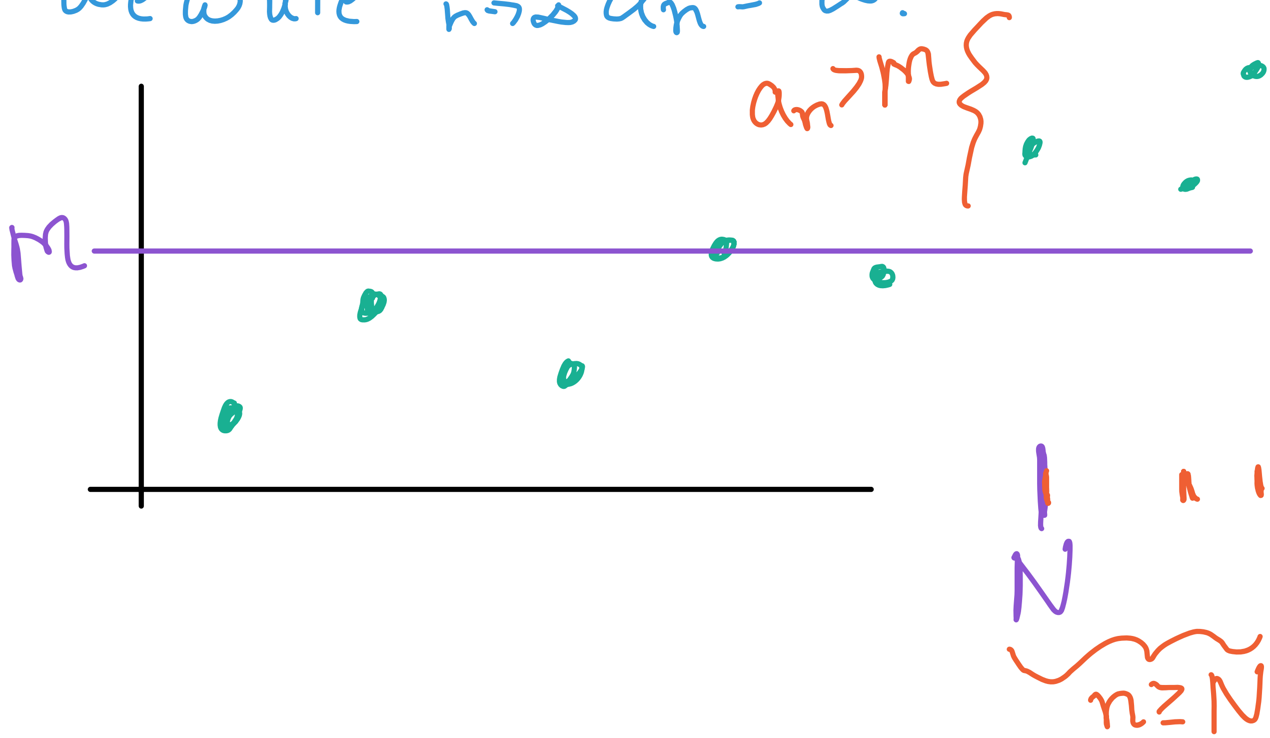
However, if a_n converges, $a_n \in \mathbb{R}$ for all but finitely many $n \in \mathbb{N}$.

Def:

- a_n diverges to $+\infty$ if, $\forall M \in \mathbb{R}$, $\exists N$ $\in \mathbb{N}$ s.t. $n \geq N$ ensures $a_n > M$.

We write $\lim_{n \to \infty} a_n = +\infty$.

- a_n **diverges to $-\infty$** if, $\forall M \in \mathbb{R}$, $\exists N$ s.t. $n \geq N$ ensures $a_n < M$. We write $\lim_{n \to \infty} a_n = -\infty$.



Q: divergence to $\pm\infty \Rightarrow$ diverges?

Def: Given a sequence a_n, the limit exists if a_n converges or a_n diverges to $\pm\infty$.

$$\Updownarrow$$

$$\lim_{n\to\infty} a_n \in \overline{\mathbb{R}}.$$

Thm (~~Squeeze~~): Suppose $a_n \le b_n$ for all but finitely many n and the limits exist. Then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

Pf: On HW4, you will show the result if both sequences converge.

If a_n diverges to $-\infty$ or b_n diverges to $+\infty$, the result is clearly true.

Suppose a_n diverges to $+\infty$. We will show b_n diverges to $+\infty$. Fix $M \in \mathbb{R}$. $\exists\, N$ s.t. $n \geq N$ ensures $M < a_n$. Choose N' so that $a_n \leq b_n \ \forall\, n \geq N'$. Let $N'' = \max\{N, N'\}$. Then $n \geq N''$ ensures $M < a_n \leq b_n$. Thus $\lim_{n\to\infty} b_n = +\infty$.

It remains to show $\lim_{n\to\infty} b_n = -\infty \implies \lim_{n\to\infty} a_n = -\infty$. See HW5.

16 Monotone Sequences and e

Recall:

increasing $a_n \le a_{n+1}$, $\forall n \in \mathbb{N}$

decreasing $a_n \ge a_{n+1}$, $\forall n \in \mathbb{N}$

monotone, if either increasing or decreasing

Rmk:

increasing sequences are bdd below as long as $a_1 \neq -\infty$

decreasing sequences are bdd above, as long as $a_1 \neq +\infty$.

Thm: All bounded monotone sequences converge.

Pf: Suppose a_n is bounded and increasing. Since a_n is bounded, $\{a_n : n \in \mathbb{N}\}$ is bounded above, its supremum exists. Let $L = \sup\{a_n : n \in \mathbb{N}\}$

Fix $\varepsilon > 0$. Since L is an upper bound, $L \geq a_n \ \forall n \in \mathbb{N}$.

Since $L-\varepsilon < L$, $L-\varepsilon$ is not an upper bound, so $\exists N$ s.t. $a_N > L-\varepsilon$. Since a_n is increasing, $a_n \geq a_N > L-\varepsilon$ for all $n \geq N$.

Thus $n \geq N$, $L-\varepsilon < a_n \leq L < L+\varepsilon$
$\Rightarrow |a_n - L| < \varepsilon$.

Now, suppose a_n is bounded and decreasing. Then $-a_n$ is bounded and increasing, so it converges to $L \in \mathbb{R}$.
Thus $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)(-a_n) = -L$.

$\square$

Ex'

Claim: If $|a|<1$, $\lim_{n\to\infty} a^n = 0$.

Pf of Claim:

If $a=0$, the result is immediate.

Suppose $0<a<1$.

1) a^n is decreasing (by induction)

2) a^n is bounded, since product of nonneg is nonneg and decreasing

Thus, $\lim_{n\to\infty} a^n = L$.

Since a^{n+1} is a subseq of a^n

$$L = \lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty} a\,a^n = aL.$$

If $L \neq 0$, then $1 = a$, which is a contradiction. Thus $L = 0$.

Suppose $-1 < a < 0$.
Then $\lim_{n\to\infty} (-a)^n = 0$. Thus

$$\lim_{n\to\infty} a^n = \lim_{n\to\infty} \underbrace{(-1)^n}_{\text{bdd}}\underbrace{(-a)^n}_{\to 0} = 0. \quad \square$$

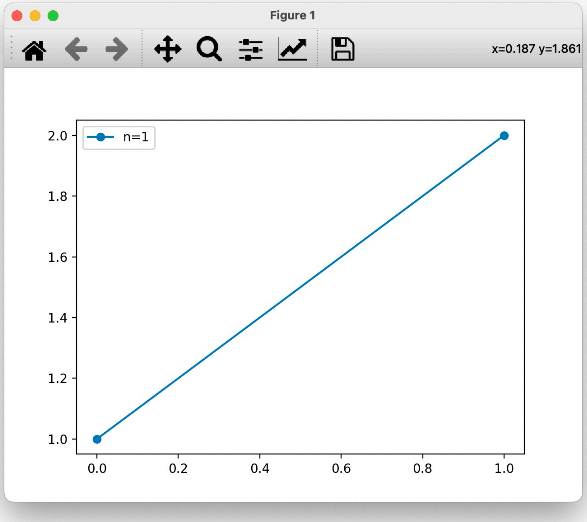


Figure 1
x=0.187 y=1.861
n=1
2.0
1.8
1.6
1.4
1.2
1.0
0.0
0.2
0.4
0.6
0.8
1.0

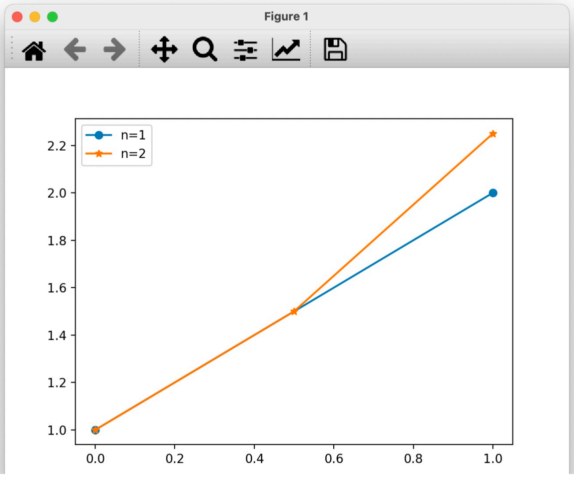
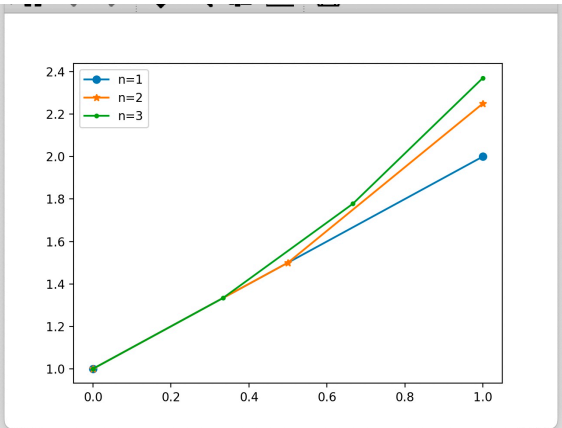


Figure 1
n=1
n=2
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.0
0.2
0.4
0.6
0.8
1.0


n=1
n=2
n=3
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.0
0.2
0.4
0.6
0.8
1.0

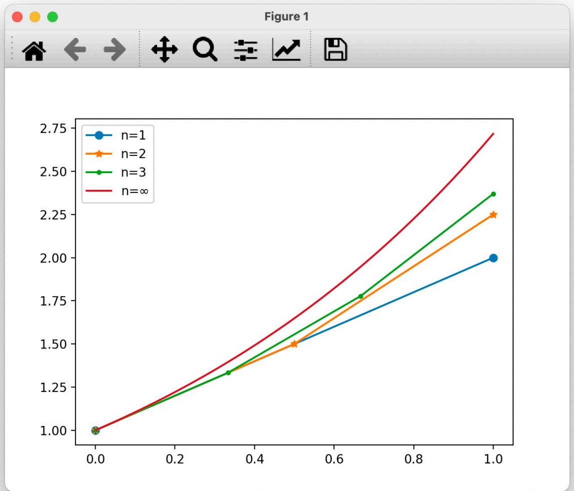


Figure 1
n=1
n=2
n=3
n=∞
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.0
0.2
0.4
0.6
0.8
1.0